Advanced System Dynamics and Control
2.151
Spring 2001
Solution Homework 6

Massachusetts Institute of Technology
Department of Mechanical Engineering

Solution 1 : Transfer functions.

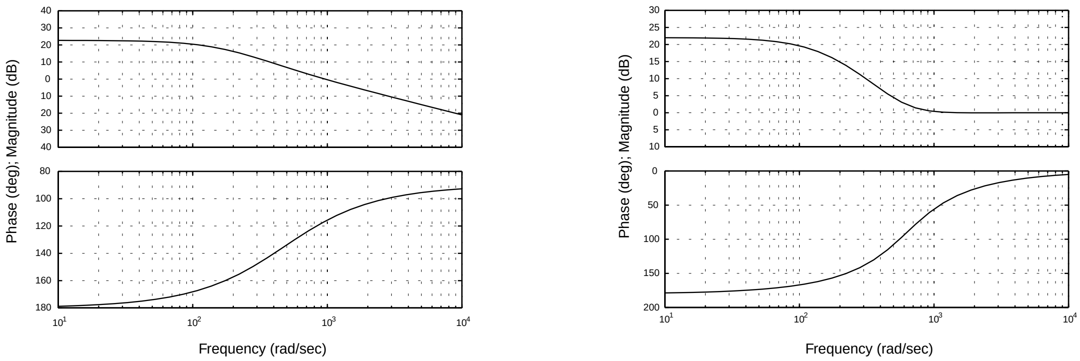
(a) The transfer functions are

$$
\begin{aligned}
\text{Forward loop TF:} \quad G(s)G_c(s) &= \frac{\beta k_d s + \beta k_p}{s^2 - \alpha} \\
&= \frac{888.425s + 426035}{s^2 - 31250} \\
\text{Return difference TF:} \quad 1 + G(s)G_c(s) &= \frac{s^2 + \beta k_d s + \beta k_p - \alpha}{s^2 - \alpha} \\
&= \frac{s^2 + 888.425s + 394785}{s^2 - 31250} \\
\text{Sensitivity TF:} \quad \frac{1}{1 + G(s)G_c(s)} &= \frac{s^2 - \alpha}{s^2 + \beta k_d s + \beta k_p - \alpha} \\
&= \frac{s^2 - 31250}{s^2 + 888.425s + 394785} \\
\text{Closed loop TF:} \quad \frac{G(s)G_c(s)}{1 + G(s)G_c(s)} &= \frac{\beta k_d s + \beta k_p}{s^2 + \beta k_d s + \beta k_p - \alpha} \\
&= \frac{888.425s + 426035}{s^2 + 888.425s + 394785}
\end{aligned}
$$

(b) (i) **Forward loop transfer function:** This transfer function has two poles and one zero. The DC gain is 13.6331. The high frequency slope is -20dB/decade. The phase starts at $-180°$ at low frequencies and reaches $-90°$ at high frequencies. The phase increases due to the zero. The forward loop transfer function bode plot is shown in Figure 1(a).

(ii) **Return difference transfer function:** This transfer function has 2 poles and 2 zeros. The gain is constant in the low and high frequency ranges. The -40 dB/decade is contributed by the poles and the $+40$ dB/decade by the zeros. The phase is $-180°$ at low frequencies because the poles contribute $-180°$ (due to the unstable pole) and there is no phase contribution by the zeros in this range. At high frequencies, the 2 zeros contribute $+180°$. This cancels out the $-180°$ contribution of the poles resulting in a zero phase. The return difference transfer function bode plot is shown in Figure 1(b).

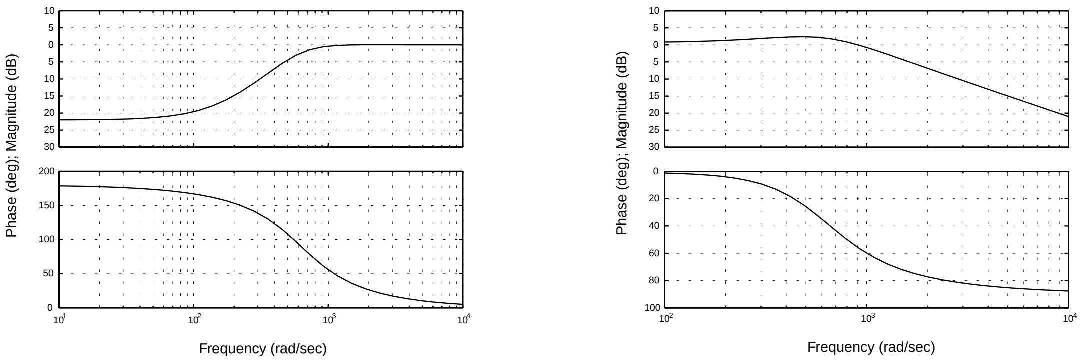
(iii) **Sensitivity transfer function:** This transfer function has 2 poles and 2 zeros. It is the inverse of the return difference transfer function. The phase is $+180°$ in the low frequency



(a) (b)

Figure 1: Magnitude and phase plots. (a) Forward loop transfer function. (b) Return difference transfer function.

range and 0 in the high frequency range. The sensitivity transfer function bode plot is shown in Figure 2(a).



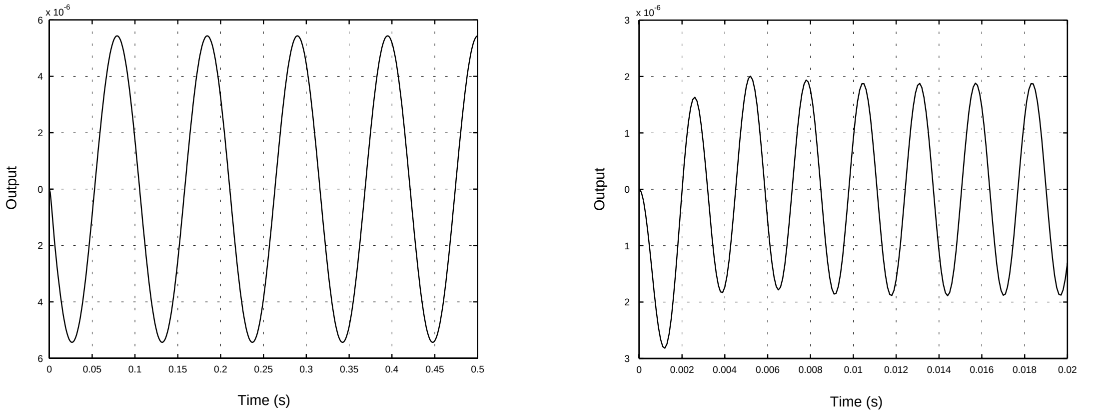
(a) (b)

Figure 2: Magnitude and phase plots (a) Sensitivity transfer function. (b) Closed loop transfer function.

(iv) **Closed-loop transfer function:** The closed loop transfer function has 2 poles and 1 zero. The magnitude plot is constant in the low frequency range and decreases at $-20\,\mathrm{dB/decade}$ in the high frequency range. The phase on the other hand is $0°$ in the low frequency range and -90 in the high frequency range. The closed-loop transfer function bode plot is shown in Figure 2(b).

(c) The system bandwidth is the frequency range $0 \leq \frac{\omega}{2\pi} \leq f_b$ where f_b is the frequency at which the gain has dropped by $3\,\mathrm{dB}$ below the DC gain. If we consider the forward loop transfer function $G(s)G_c(s)$, its gain must be high in the system bandwidth to ensure good command following and disturbance rejection. From the forward loop transfer function bode plot, $f_b = 119.3668\,\mathrm{rad/s} = 18.9987\,\mathrm{Hz}$.

(d) The time response of the system to the given noise inputs are given in Figure 3. For the noise with frequency of $\frac{f_b}{2}$, the gain of the closed loop transfer function is approximately $1\,\mathrm{dB}$, consequently, the noise is amplified by $1\,\mathrm{dB}$. For the noise with frequency of $20f_b$, the gain of the closed loop transfer function is approximately $-10\,\mathrm{dB}$, consequently, the noise is amplified by $-10\,\mathrm{dB}$.



(a) (b)

Figure 3: System time response to time inputs. (a) $N(s) = N_o \sin(\pi f_b t)$. (b) $N(s) = N_o \sin(40\pi f_b t)$

(e) The zero z_o introduced by the PD controller is found by

$$\begin{aligned} k_p + k_d s &= 0 \\ \Rightarrow \quad s &= -\frac{k_p}{k_d} = -479.539\,\mathrm{rad/s} \end{aligned}$$

The effects of the zero on the system response are shown in Figures 4 to 6. In this case, we have changed the location of the zero by changing k_p. Note that when the zero is closer to the origin, the time response slows down for this system.

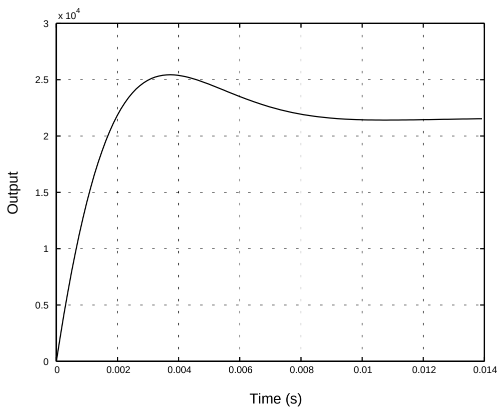


Figure 4: Effects of the zero on the system time response — original response.

Solution 2 : Transfer functions: A drive system.

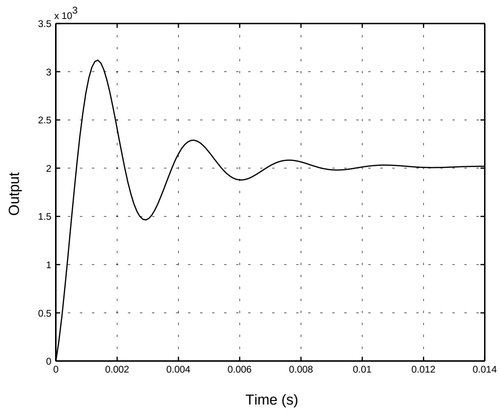


Figure 5: Effects of the zero on the system time response — $|z_1| = 10|z_o|$.

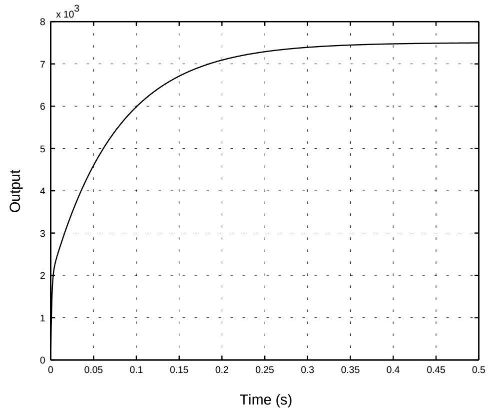


Figure 6: Effects of the zero on the system time response — $|z_2| = .1|z_o|$.

(a) Using the given system parameters — gear ratio, $r = 0.5$, motor inertia $I_m = 0.08\,\text{ft.lb.s}^2.\text{rad}^{-1}$, load inertia $I_l = 0.8\,\text{ft.lb.s}^2.\text{rad}^{-1}$ and $R = 0.26\,\text{ft.lb.s.rad}^{-1}$, the effective system inertia I reflected on the motor axis is

$$\begin{aligned} I &= I_m + I_l r^2 \\ &= 0.08 + 0.08 + 0.08(0.25) \\ &= 0.28\,\text{ft.lb.s}^2.\text{rad}^{-1} \end{aligned} \tag{1}$$

(b) The transfer function $\frac{\Omega(s)}{U(s)}$ is given by

$$\begin{aligned} \frac{\Omega(s)}{U(s)} &= \frac{1/I}{s + R/I} \\ &= \frac{3.5714}{s + 0.9257} \end{aligned} \tag{2}$$

(c) The magnitude and phase are shown in Figure 7. This is a first order system. At low frequencies, the output velocity will follow the command input with a small phase lag. The DC gain (the gain at zero frequency) is $\frac{1/I}{R/I} = \frac{1}{R} = 11.7005\,\text{dB}$. For inputs with high frequencies, $\omega \gg \omega_b$, the output amplitude will drop significantly. In fact, it will behave as $|G(j\omega)| \approx \left|\frac{1/I}{j\omega}\right| = \frac{1}{I\omega}$. This is also equal to

$$\begin{aligned} 20\log_{10}|Gj\omega| &\approx -20\log_{10}(I\omega) \\ &= -11.0568 - 10\log_{10}(\omega) \end{aligned} \tag{3}$$

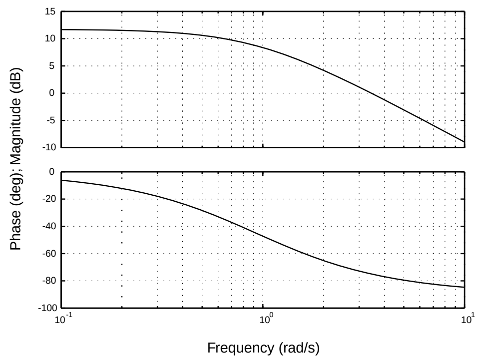


Figure 7: Magnitude and phase of $\frac{\Omega(s)}{U(s)}$.

On a logarithmic scale, the gain resembles a $-20\,\text{dB/decade}$ line. At high frequencies, the phase lag will be deg -90. This is shown by

$$\begin{aligned} \arg[G(j\omega)] &\approx \arg\left[\frac{1/I}{j\omega}\right] \\ &= \arg\left[\frac{-j}{I\omega}\right] \end{aligned} \tag{4}$$

The break frequency for this first order system occurs at the point where the phase is deg -45, i.e. $f_b = 0.1473\,\text{Hz}$.

(d) The time response of the system for a step input of $u(t) = 95\,\text{ft.lb}$ is shown in Figure 8. The time constant is $1.077\,\text{s}$ and the settling time is $4\tau = 4.308\,\text{s}$.

(e) The time response of the system with sinusoidal inputs is shown in Figures 9 to 11. The frequency of the input $u_1(t) = 10\sin(0.2\pi f_b t)$ is 10 times lower than the break frequency so the phase lag should be approximately deg -6 and the gain between the output and the input is approximately $|G(j\omega)| = R^{-1} = 3.858$. For our input amplitude of 10, the amplitude of the output should be ≈ 38.58. The frequency of the input $u_2(t) = 10\sin(2\pi f_b t)$ is equal to the break frequency, so the phase lag should be deg -45 and the gain between the output and the input is

$$\begin{aligned} |G(2\pi f_b j)| &= \left|\frac{1/I}{j\omega + R/I}\right| \\ &= 2.7241 \end{aligned} \tag{5}$$

The output amplitude should be approximately $2.741 \times 10 = 27.241$. The frequency of the input $u_3(t) = 10\sin(20\pi f_b t)$ is 10 times higher than the break frequency, so the phase lag should be

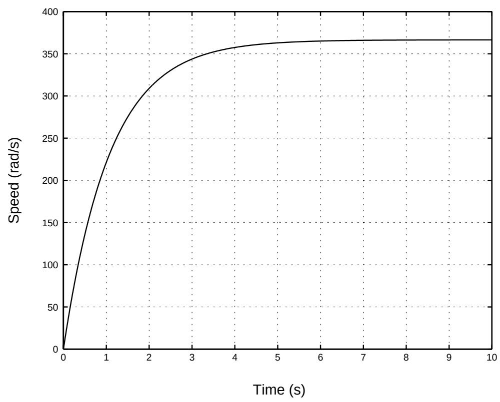


Figure 8: Time response of system for step input of $u(t) = 95\,\mathrm{ft.lb}$.

approximately deg 84 and the gain between the output and input is

$$\begin{aligned} |G(20\pi f_b j)| &= \left|\frac{1/I}{I\omega}\right| \\ &= 0.3858 \end{aligned} \tag{6}$$

Thus, the output amplitude should be approximately $0.3858 \times 10 = 3.858$. These three cases indicate that the gain starts to decrease rapidly for frequencies higher than the break frequency.

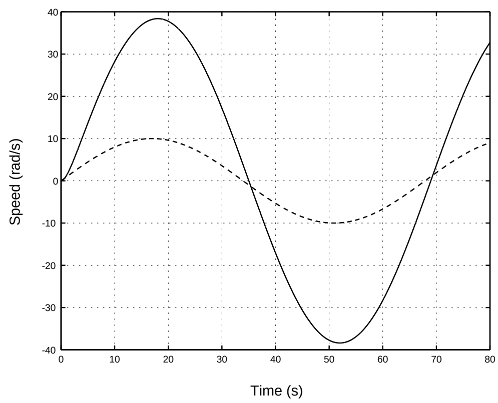


Figure 9: Time response of system for $u_1(t) = 10\sin(0.2\pi f_b t)$.

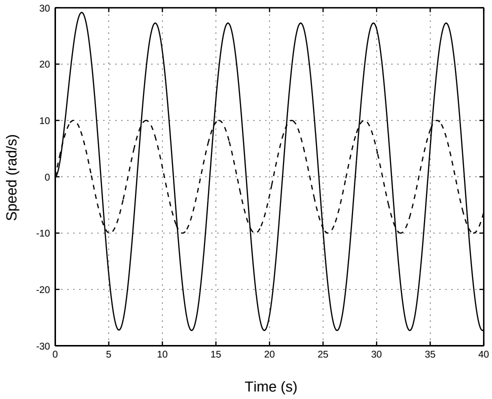


Figure 10: Time response of system for $u_2(t) = 10\sin(2\pi f_b t)$.

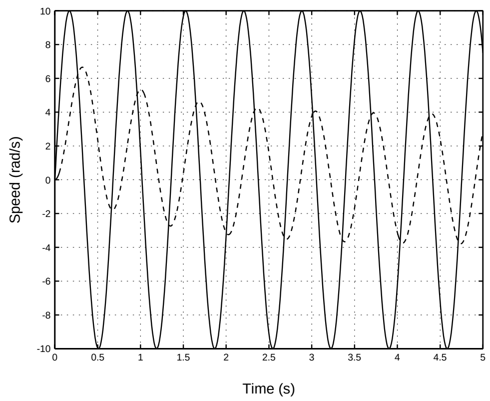


Figure 11: Time response of system for $u_1(t) = 10\sin(20\pi f_b t)$.