PS#2 Solutions
2.717J Fall 2000

#1

a) Power of Thin Lens $\quad \phi = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)$ $\quad$ (3.59)

$$\Delta\phi = \Delta n\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)$$

$$= \underbrace{\frac{\Delta n}{n-1}}_{V}\left[n-1\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\right] \phi$$

$$= \frac{1}{V}\phi = \frac{1}{V}\frac{1}{f}$$

b) $\phi = \phi_1 + \phi_2 - t\,\phi_1\phi_2$ $\quad$ (3.79a)

$$\Delta\phi = \Delta\phi_1 + \Delta\phi_2 - t\left[\phi_1\Delta\phi_2 + \phi_2\Delta\phi_1\right]$$

$$= \frac{1}{V_1 f_1} + \frac{1}{V_2 f_2} - t\left[\frac{1}{f_1 V_2 f_2} + \frac{1}{f_2 V_1 f_1}\right] = 0 \qquad (*)$$

$$\boxed{t = \frac{\frac{1}{V_1 f_1} + \frac{1}{V_2 f_2}}{\frac{1}{f_1 f_2}\left(\frac{1}{V_1} + \frac{1}{V_2}\right)} = \frac{V_1 f_1 + V_2 f_2}{V_1 + V_2}}$$

$$\boxed{\phi = \frac{1}{V_1 + V_2}\left(\frac{V_1}{f_1} + \frac{V_2}{f_2}\right)}$$

identical lenses $\quad V_1 = V_2 = V$
$\quad f_1 = f_2 = f$

$$t = \frac{2Vf}{2V} = f$$

$$\phi = 1/f$$

c) eq (*) w/ $t=0 \Rightarrow \frac{1}{V_1 f_1} + \frac{1}{V_2 f_2} = 0 \Rightarrow \boxed{\frac{1}{f_2} = -\frac{V_2}{V_1 f_1}}$

$$\& \quad \boxed{\phi = \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_1} - \frac{V_2}{V_1}\frac{1}{f_1} = \frac{1}{f_1}\left(1 - \frac{V_2}{V_1}\right)}$$

d) design for $\underline{\phi = 100}$ (Note $D \equiv$ Diopter $\equiv \left[\frac{1}{m}\right]$)

$\phi = \frac{1}{f_1}\left(1 - \frac{V_1}{V_2}\right) = 10$

(1) BaLF4 $V_D = 53.75$ $n = 1.58$

(2) SF58 V_D 21.50 $n = 1.98$

$\underline{f_1} = \frac{1}{10}\left(1 - \frac{21.50}{53.75}\right) = \underline{.06m} \Rightarrow \underline{\phi_1 = 16.66 m^{-1}}$

$\underline{\phi_2} = \frac{1}{f_2} = \frac{-V_2}{V_1 f_1} = \frac{-21.50}{53.75(.06)} = \underline{-6.666 m^{-1}} \Rightarrow \underline{f_2 = -.15 m}$

$\phi_1 = (n_1 - 1)\left(\frac{1}{R_1} - \frac{1}{R_1'}\right) = (.58)\left(\frac{1}{R_1} - \frac{1}{R_1'}\right) = 16.66$

• $\left(\frac{1}{R_1} - \frac{1}{R_1'}\right) = 28.736$

$\phi_2 = (n_2 - 1)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right) = (.98)\left(-\frac{1}{R_1'} - \frac{1}{R_2'}\right) = -6.66$

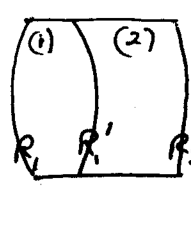
• $\frac{1}{R_1'} + \frac{1}{R_2'} = 6.803$

we have to set $\underline{R_1}$, $\underline{R_1'}$ & $\underline{R_2'}$

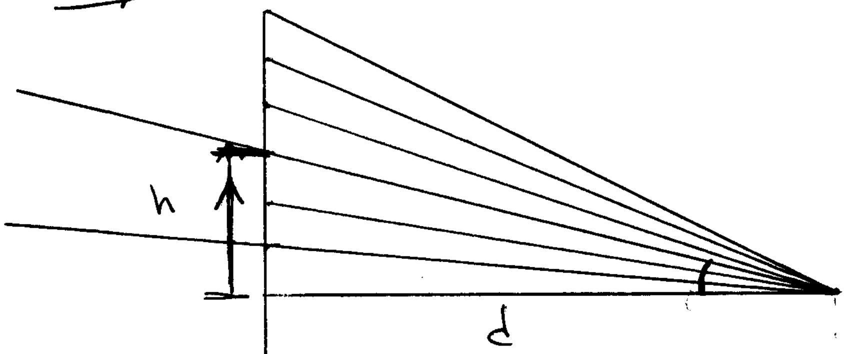
let $R_1 = R_1' \Rightarrow \frac{2}{R_1} = 28.736 \Rightarrow R_1 = .0696m$

$R_1' = -.0696m$

$R_2' = \frac{+1}{-\frac{1}{R_1'} + 6.803} = -.133m$

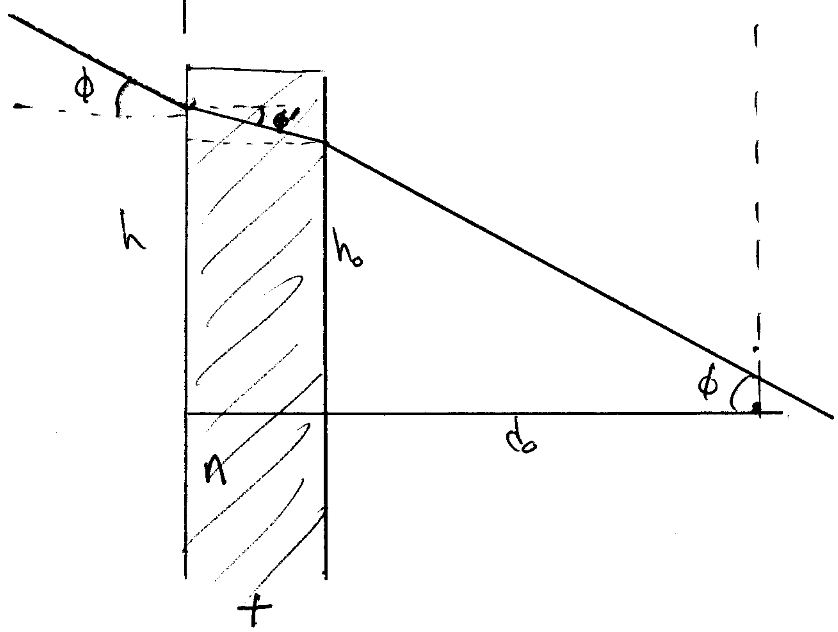


#2/



$$\tan\phi = \frac{h}{d}$$

$$\sin\phi = \frac{h}{\sqrt{h^2+d^2}} = \frac{(h/d)}{\sqrt{1+(\frac{h}{d})^2}}$$



$$\tan\phi = \frac{h_o}{d_o} = \frac{h}{d}$$

$$d_o = \frac{h_o d}{h}$$

where $h_o = h - t\tan\phi'$

$$d_o = \left(\frac{h - t\tan\phi'}{h}\right) d$$

from snell $n\sin\phi' = \sin\phi \Rightarrow \sin\phi' = \frac{1}{n}\frac{(h/d)}{\sqrt{1+(h/d)^2}}$

we want to determine d_o as a function of height

we have a term $\tan\phi'$ in The equation For d_o

From snell we have $\sin\phi'$

From geometry we have $\cos\phi' = \sqrt{1-\sin^2\phi'}$

so linearize + 2nd order The equation For d_o

Note: $(1-x)^{1/2} \cong 1 + \frac{1}{2}(1-x)^{-1/2}(-1)\Big|_{x\approx 0}\frac{\Delta x}{1!} + \frac{-1}{4}(1-x)^{-3/2}(-1)(-1)\Big|_{x\approx 0}\frac{\Delta x^2}{2!}$

$$\approx 1 - \frac{1}{2}\Delta x - \frac{1}{8}\Delta x^2 \cdots$$

$$(1-x)^{-1/2} \approx 1 + \left(\frac{-1}{2}\right)(1-x)^{-3/2}(-1)\Big|_{x\approx 0}\frac{\Delta x}{1!} + \left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right)(1-x)^{-5/2}(-1)\Big|_{x\approx 0}\frac{\Delta x^2}{2!}$$

$$\approx 1 + \frac{1}{2}\Delta x - \frac{3}{8}\Delta x^2 \cdots$$

$$\sin x \approx x - \frac{x^3}{6} + \cdots$$

$$\cos x \approx 1 - \frac{x^2}{2} + \cdots$$

$$\sin\phi' = \frac{1}{n}\frac{h/d}{\sqrt{1+(h/d)^2}} = \frac{1}{n}\frac{h}{d}\left(1+\left(\frac{h}{d}\right)^2\right)^{-1/2} \approx \frac{1}{n}\frac{h}{d}\left(1-\frac{1}{2}\left(\frac{h}{d}\right)^2\right)$$

$$\frac{1}{\cos\phi'} = \frac{1}{\sqrt{1-\sin^2\phi'}} \approx 1 + \frac{1}{2}\sin^2\phi' \approx 1 + \frac{1}{2}\frac{1}{n^2}\left(\frac{h}{d}\right)^2$$

$$\tan\phi' = \frac{\sin\phi'}{\cos\phi'} \approx \frac{1}{n}\frac{h}{d}\left(1-\frac{1}{2}\left(\frac{h}{d}\right)^2\right)\left(1+\frac{1}{2n^2}\left(\frac{h}{d}\right)^2\right)$$

$$\approx \frac{1}{n}\frac{h}{d}\left(1 + \left[\frac{1}{2n^2}\left(\frac{h}{d}\right)^2 - \frac{1}{2}\left(\frac{h}{d}\right)^2\right]\right)$$

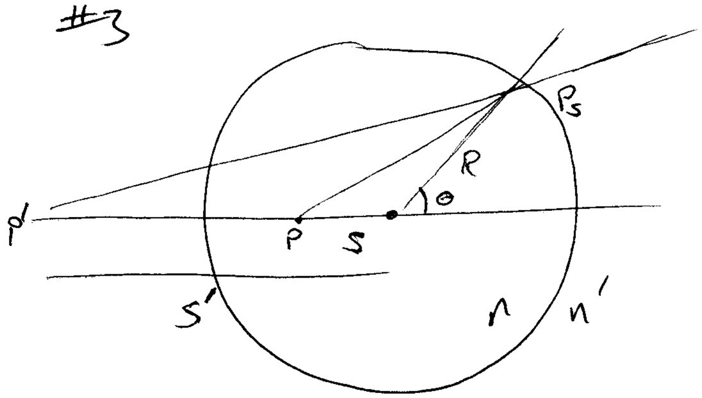
$$\approx \frac{1}{n}\frac{h}{d}\left(1 + \frac{1}{2}\left(\frac{h}{d}\right)^2\left(\frac{1}{n^2} - 1\right)\right)$$

$\therefore$

$$d_0 \cong \left(1 - \frac{t}{h}\tan\phi'\right)d = \left[1 - \frac{t}{h}\frac{1}{n}\frac{h}{d}\left(1 + \frac{1}{2}\left(\frac{h}{d}\right)^2\left(\frac{1}{n^2}-1\right)\right)\right]d$$

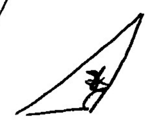
$$\boxed{d_0 = \underbrace{\left(d - \frac{t}{n}\right)}_{\text{paraxial}} + \underbrace{t\left(\frac{1}{2}\right)\left(\frac{h}{d}\right)^2\left(-\frac{1}{n^3} + \frac{1}{n}\right)}_{\text{spherical aberration}}}$$

#3



→ Prove spherical surface is Cartesian for refraction if $n^2 s = nn'R = n'^2 s'$

Law of cosines



$a^2+b^2-2ab\cos * = c^2$

$$OPL(PP') = OPL(P P_s) - OPL(P_s P')$$
$$= n\,\overline{PP_s} - n'\,\overline{P_s P'}$$

$$OPL(PP') = n\,(s^2+R^2+2RS\cos\theta)^{1/2} - n'(S'^2+R^2+2RS'\cos\theta)^{1/2}$$

show
$$\frac{d}{d\theta} OPL(PP') = 0 \quad ..$$

or work backwards & show that $OPL(PP') = \text{constant}$ if we substitute $n^2 S = n'^2 S' = nn'R$

$$(n\, OPL(PP_s))^2 = n^2 s^2 + n^2 R^2 + 2n^2 RS \cos\theta$$

| using $n^2 s = nn'R$

$s = \frac{n'}{n} R$

$$(n\, OPL(PP_s))^2 = n'^2 R^2 + n^2 R^2 + 2R\, nn' R \cos\theta$$

$$(n\, OPL(P'P_s))^2 = n'^2 s'^2 + n'^2 R^2 + 2n'^2 RS' \cos\theta$$

| using $n'^2 s' = nn'R$

$s' = \frac{n}{n'} R$

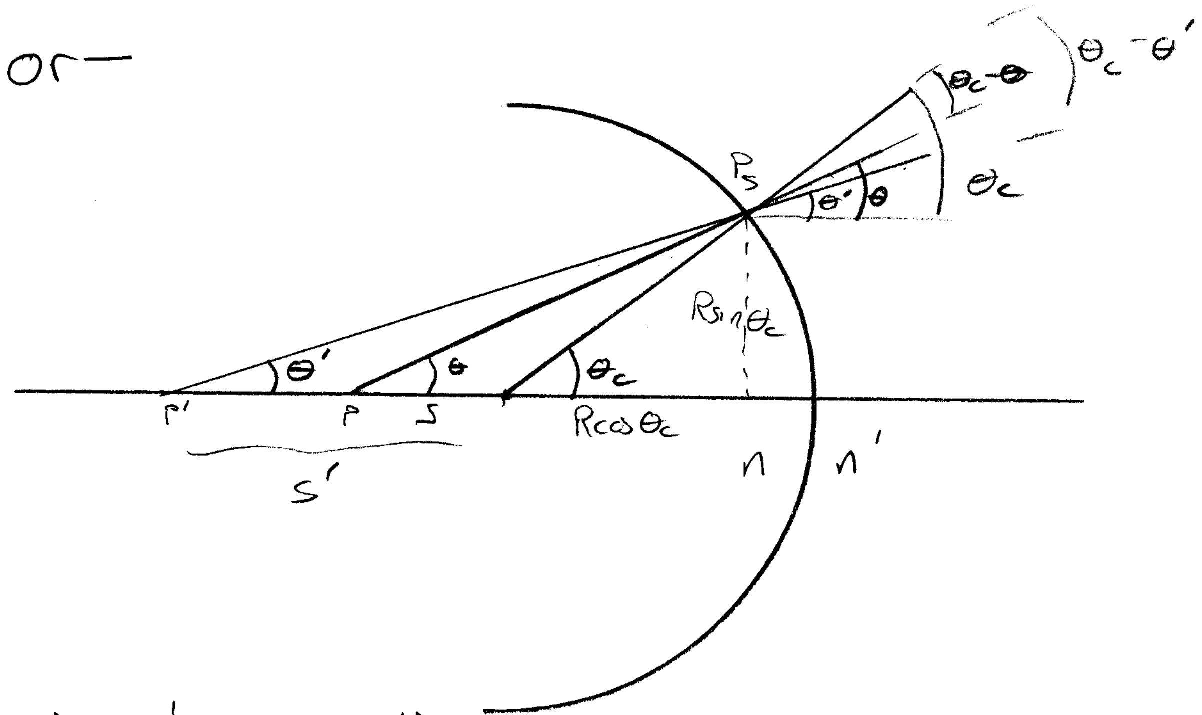
$$(n\, OPL(P'P_s))^2 = n^2 R^2 + n'^2 R^2 + 2R\, nn' R \cos\theta$$

$\therefore$ optical paths are same

and

$OPL(PP') = \text{constant}$ if $n^2 s = nn'R = n'^2 s'$

#3/ -or-



SNELL

$$n\sin(\theta_c-\theta) = n'\sin(\theta_c-\theta')$$

$$n(\sin\theta_c\cos\theta - \cos\theta_c\sin\theta) = n'(\sin\theta_c\cos\theta' - \cos\theta_c\sin\theta')$$

$$\cos\theta = \frac{s + R\cos\theta_c}{\overline{PP_s}} \qquad \cos\theta' = \frac{s' + R\cos\theta_c}{\overline{P'P_s}}$$

$$\sin\theta = \frac{R\sin\theta_c}{\overline{PP_s}} \qquad \sin\theta' = \frac{R\sin\theta_c}{\overline{P'P_s}}$$

$$n\left(\sin\theta_c\frac{s+\cancel{R\cos\theta_c}}{\overline{PP_s}} \cancel{-\cos\theta_c\frac{R\sin\theta_c}{\overline{PP_s}}}\right) = n'\left(\sin\theta_c\frac{s'+\cancel{R\cos\theta_c}}{\overline{P'P_s}} \cancel{-\cos\theta_c\frac{R\sin\theta_c}{\overline{P'P_s}}}\right)$$

$$n\left(\sin\theta_c\frac{s}{\overline{PP_s}}\right) = n'\left(\sin\theta_c\frac{s'}{\overline{P'P_s}}\right)$$

$$\frac{ns}{\overline{PP_s}} = \frac{n's'}{\overline{P'P_s}}$$

$(\times nn')$

$$n^2 s' = n'nR$$

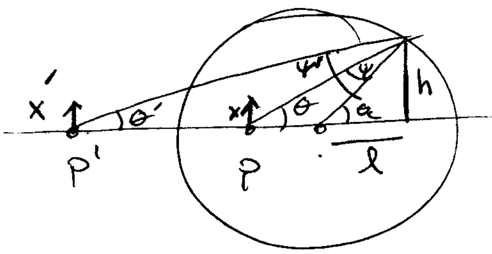
$$\frac{nn'ns}{\overline{PP_s}} = \frac{nn'n's'}{\overline{P'P_s}}$$

$n'^2 s = nn'R$

$$\frac{n'nn'R}{\overline{PP_s}} = \frac{nnn'R}{\overline{P'P_s}} \Rightarrow \frac{n'}{\overline{PP_s}} = \frac{n}{\overline{P'P_s}} \Rightarrow \boxed{n'\overline{P'P_s} = n\overline{PP_s}}$$

OPL's are equal

#4/ Now show P & P' form aplanatic pair.
That is show sine condition is obeyed.



show $n'x'\sin\theta' = nx\sin\theta$

-or-

$$\frac{\sin\theta'}{\sin\theta} = \text{const}$$

Snell

$$n\sin\psi = n'\sin\psi'$$

$$n\sin(\theta_c - \theta) = n'\sin(\theta_c - \theta')$$

$$n\left[\sin\theta_c\cos\theta - \sin\theta\cos\theta_c\right] = n'\left[\sin\theta_c\cos\theta' - \sin\theta'\cos\theta_c\right]$$

$$n\left[\frac{h}{R}\frac{s+\cancel{l}}{\overline{PP_s}} - \frac{h}{\overline{PP_s}}\frac{\cancel{l}}{R}\right] = n'\left[\frac{h}{R}\frac{s'+\cancel{l}}{\overline{P'P_s}} - \frac{h}{\overline{P'P_s}}\frac{\cancel{l}}{R}\right]$$

$$n\left[\frac{hs}{R\overline{PP_s}}\right] = n'\left[\frac{hs'}{R\overline{P'P_s}}\right]$$

$$\boxed{\frac{\overline{PP_s}}{\overline{P'P_s}} = \frac{ns}{n's'} \overset{\times\frac{n'n}{n'n}}{=} \frac{nn'ns}{n'n'ns'} \overset{n^2s=n'nR,\ n'^2s'=n'nR}{=} \frac{n'nn'R}{nnn'R} = \frac{n'}{n}}$$

Sine condition

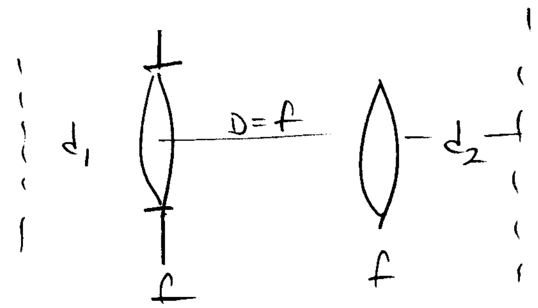
$$nx\sin\theta = n'x'\sin\theta'$$

snell result w/ $n^2s = n'^2s' = nn'R$

$$\frac{nx}{n'x} = \boxed{\frac{\sin\theta'}{\sin\theta}} = \frac{h/\overline{PP_s}}{h/\overline{P'P_s}} = \frac{\overline{P'P_s}}{\overline{PP_s}} = \boxed{\frac{n}{n'} = \text{constant}}$$

so sine condition is satisfied
& is cartesian (#3) so ∴ aplanatic

#6



a)
$$\begin{pmatrix}\alpha\\ x\end{pmatrix} = \begin{pmatrix}1&0\\ d_2&1\end{pmatrix}\begin{pmatrix}1&-1/f\\ 0&1\end{pmatrix}\begin{pmatrix}1&0\\ D&1\end{pmatrix}\begin{pmatrix}1&-1/f\\ 0&1\end{pmatrix}\begin{pmatrix}1&0\\ d_1&1\end{pmatrix}\begin{pmatrix}\alpha\\ x\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\ d_2&1\end{pmatrix}\begin{pmatrix}1-D/f & -\frac{1}{f}+\frac{1}{f}\left(-\frac{D}{f}+1\right)\\ D & -\frac{D}{f}+1\end{pmatrix}\begin{pmatrix}1&0\\ d_1&1\end{pmatrix}\begin{pmatrix}\alpha\\ x\end{pmatrix}$$

using $D = F$

$$= \begin{pmatrix}1&0\\ d_2&1\end{pmatrix}\begin{pmatrix}0&-1/f\\ f&0\end{pmatrix}\begin{pmatrix}1&0\\ d_1&1\end{pmatrix}$$

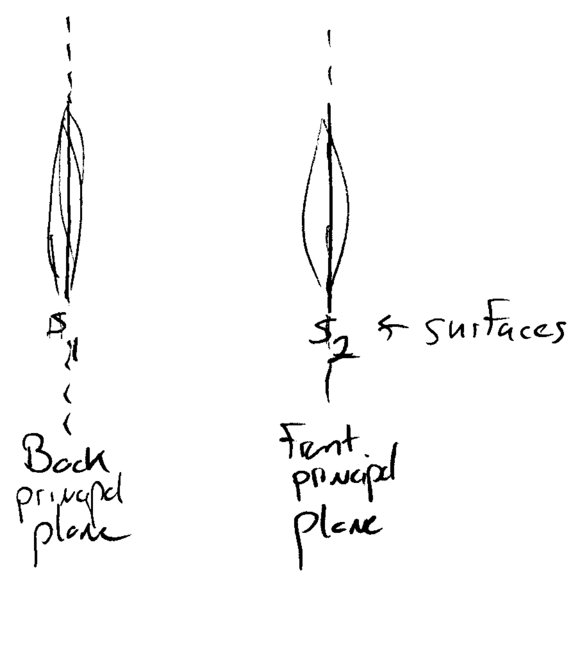
$$= \begin{pmatrix}-d_1/f & -1/f\\ -d_1d_2/f + f & -d_2/f\end{pmatrix}$$

$=1$

$d_1 = -f$

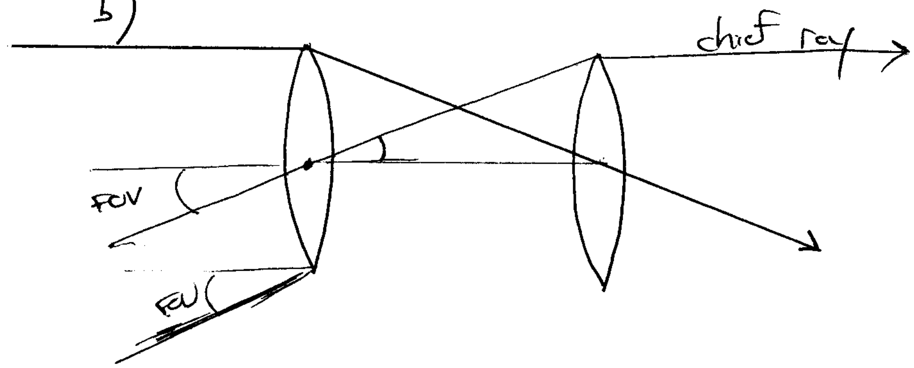
$\boxed{\text{Focal length} = f}$

$=1 \Rightarrow d_2 = -f$



$\boxed{BFL = 0}$

b)

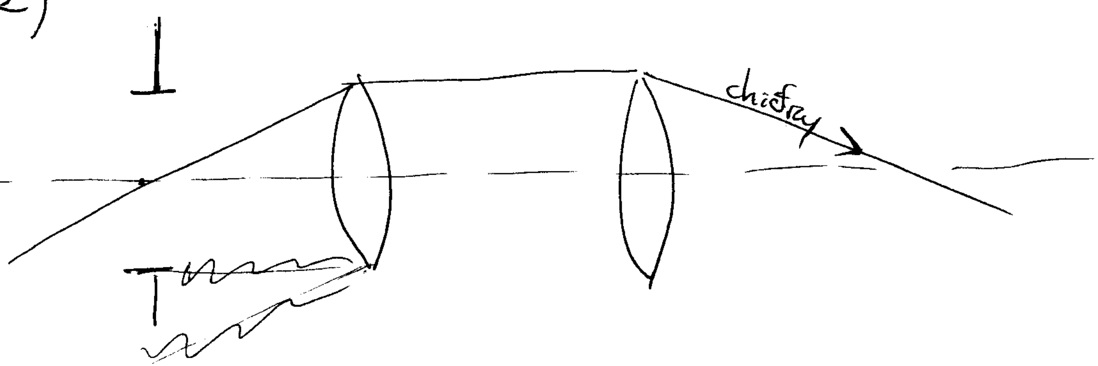


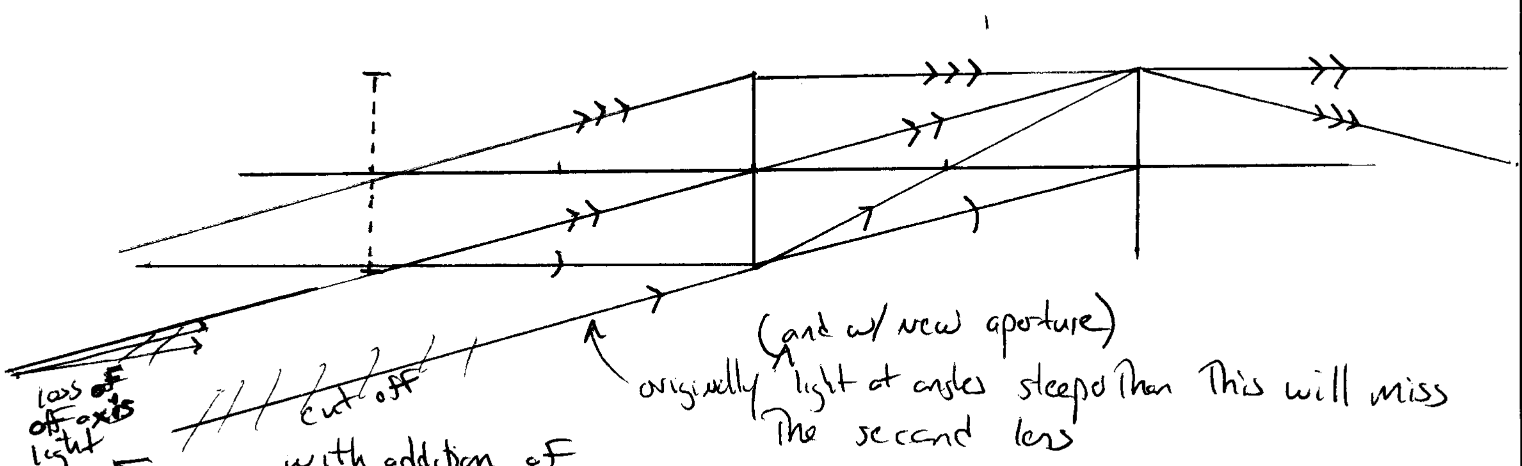
c) FOV (measured in angular subtense) is given by

$$FOV = \arctan\left(\frac{1/2 \text{ aperture}}{\text{focal length}}\right) \approx \pm 6^\circ$$

d) $$F/\# \equiv \frac{EFL}{\text{Aperture}} = 5/1 = 5$$

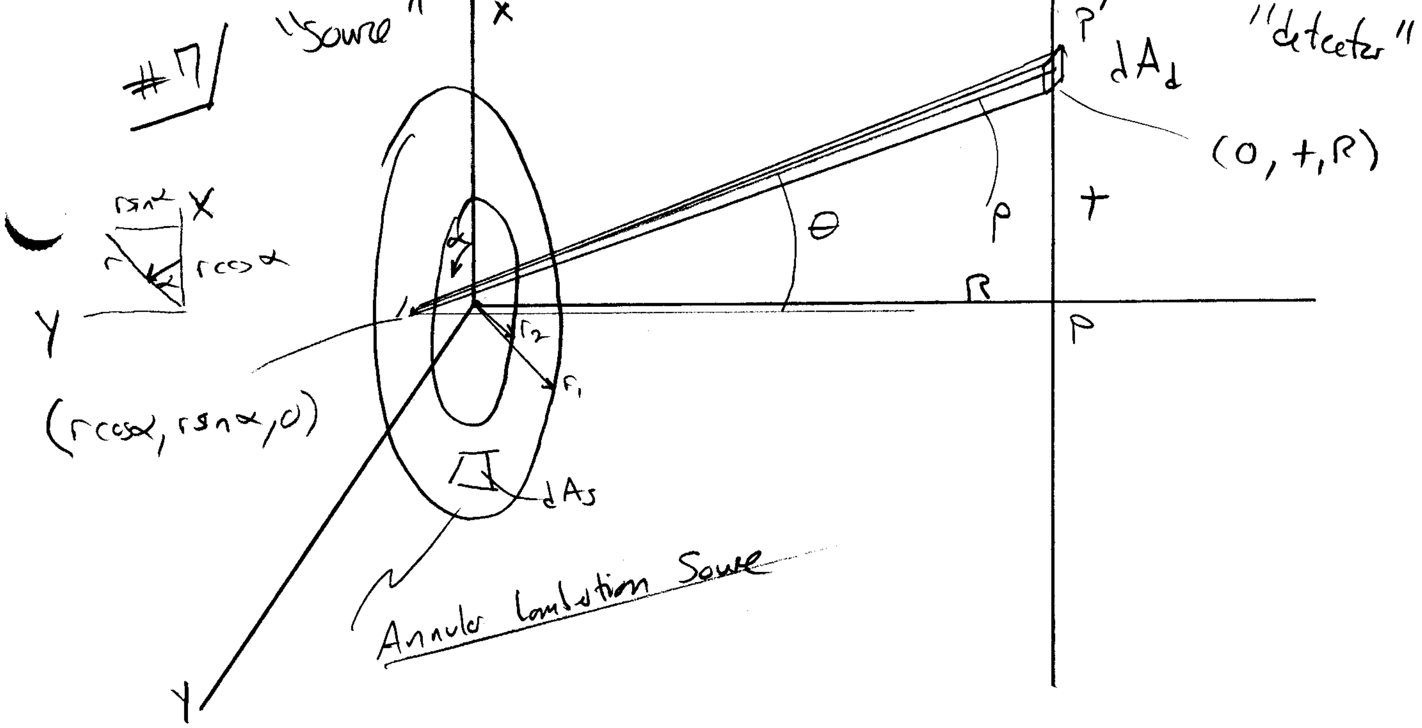
e)

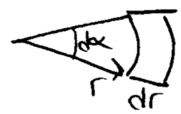




(and w/ new aperture)
originally light at angles steeper than this will miss the second lens

with addition of aperture at f "in front"
we'll get less light through the system — <u>vignetting</u>



source Area dA_s =  $r\,d\alpha$ $= r\,dr\,d\alpha$

Flux at the detector

$$d^2\phi = L\, d\Omega\, dA_s \cos\theta$$

Note: $d\Omega$ is the solid angle that an elemental area dA_d subtends as seen from a point on the source.

$$d\Omega = \frac{dA_d \cos\theta}{\rho^2}$$

From geometry

$$\rho^2 = (r\cos\alpha - 0)^2 + (r\sin\alpha - t)^2 + (0 - R)^2$$
$$= r^2\cos^2\alpha + r^2\sin^2\alpha - 2rt\sin\alpha + t^2 + R^2$$
$$= r^2 + t^2 + R^2 - 2rt\sin\alpha$$

so Flux is given

$$d^2\phi = L\underbrace{\left(\frac{dA_d\cos\theta}{r^2+t^2+R^2-2rt\sin\alpha}\right)}_{d\Omega}\underbrace{\left(r\,dr\,d\alpha\right)}_{dA_s}\cos\theta$$

total Flux reaching dA_d

$$d\phi = \int_{\substack{1/2 \\ \text{solid} \\ \text{sphere}}} d^2\phi = \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} dr L \left(\frac{dA_d \cos\theta}{r^2+t^2+R^2-2rt\sin\alpha} \right) \left(r \quad \right) \cos\theta$$

$$\cos\theta = \frac{R}{\rho}$$

$$\cos^2\theta = \frac{R^2}{\rho^2} = \frac{R^2}{r^2+t^2+R^2-2rt\sin\alpha}$$

$$d\phi = dA_d \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} dr \, L \frac{r R^2}{(r^2+t^2+R^2-2rt\sin\alpha)^2}$$

The irradiance @ the detector:

$$E \equiv \frac{d\phi}{dA_d} = L R^2 \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} \frac{r}{(r^2+t^2+R^2-2rt\sin\alpha)^2}$$

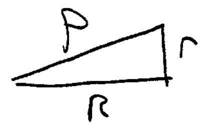
Check//

$$E = LR^2 \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} dr \frac{r}{(r^2+t^2+R^2-2rt\sin\alpha)^2}$$

if $t=0$
$r_2=0$

$$E = LR^2 \int_0^{2\pi} d\alpha \int_0^{r_1} \frac{dr\ r}{(r^2+R^2)^2}$$

note $2r\,dr = d(r^2) = d(\rho^2)$
$\therefore r\,dr = \frac{1}{2} d(\rho^2)$

why 

$\rho^2 = r^2 + R^2$
$d(\rho^2) = d(r^2) + 0$

$$E = LR^2\, 2\pi \int_{R^2}^{R^2+r_1^2} \frac{\frac{1}{2} d(\rho^2)}{(\rho^2)^2}$$

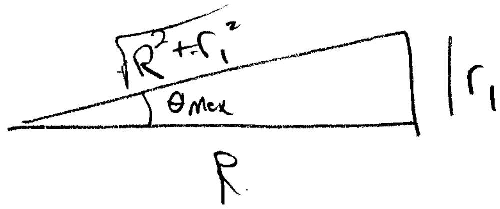
$$\int_a^b \frac{dx}{x^2} = -\frac{1}{x}\Big|_a^b = \frac{1}{a} - \frac{1}{b}$$

$$\boxed{= \pi L R^2 \left(\frac{1}{R^2} - \frac{1}{R^2+r_1^2}\right)}$$

$$= \pi L \left(1 - \frac{R^2}{R^2+r_1^2}\right)$$

$$= \pi L \left(\frac{R^2+r_1^2-R^2}{R^2+r_1^2}\right)$$

$$= \pi L \left(\frac{r_1^2}{R^2+r_1^2}\right)$$

note 

$\sin\theta_{max} = \frac{r_1}{\sqrt{r_1^2+R^2}}$

$\sin^2\theta_{max} = \frac{r_1^2}{r_1^2+R^2}$

$$\boxed{E = \pi L \sin^2\theta_{max}}$$

part a/ $t=0$

$$E = LR^2 \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} dr \frac{r}{(r^2+t^2+R^2-2rt\sin\alpha)^2}\Bigg|_{t=0}$$

$$= LR^2 \int_0^{2\pi} d\alpha \int_{r_2}^{r_1} \frac{dr\ r}{(r^2+R^2)^2}$$

$$= LR^2 2\pi \int_{r_2}^{r_1} \frac{dr\ r}{(r^2+R^2)^2}$$

again note

$rdr = \frac{1}{2} d(\rho^2)$

$\rho^2 = R^2 + r^2$

$$= LR^2 2\pi \int_{R^2+r_2^2}^{R^2+r_1^2} \frac{\frac{1}{2} d(\rho^2)}{(\rho^2)^2}$$

$$= LR^2 \pi \left[\frac{1}{R^2+r_2^2} - \frac{1}{R^2+r_1^2}\right]$$

$$= LR^2 \pi \left[\frac{R^2+r_1^2-R^2-r_2^2}{(R^2+r_2^2)(R^2+r_1^2)}\right]$$

$$E = LR^2 \pi \left[\frac{r_1^2-r_2^2}{(R^2+r_2^2)(R^2+r_1^2)}\right]$$

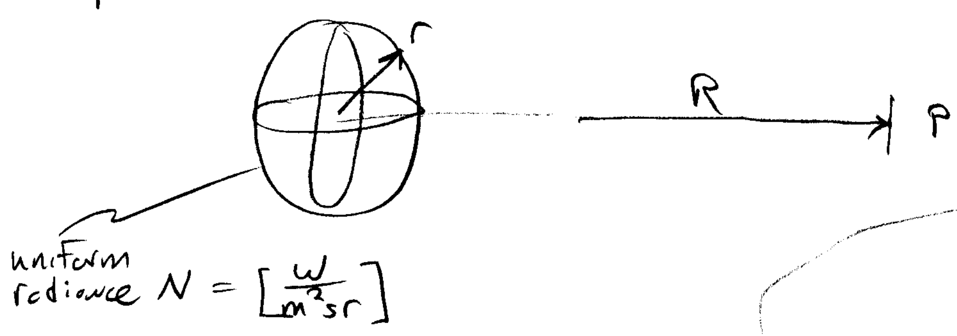
$r_1 = 10\ cm$

$r_2 = 5\ cm$ $\Rightarrow$ $\underline{E = .023\ W/m^2 str}$

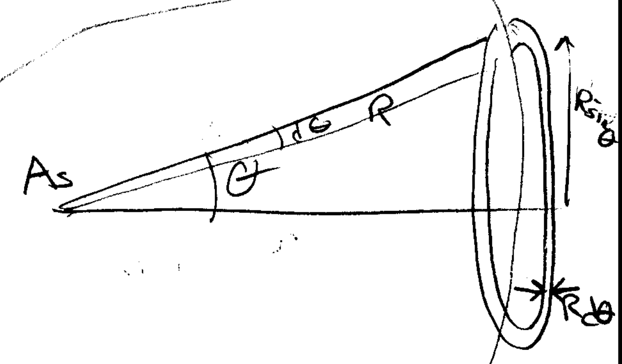
$R = 1\ m$

$L = 1\ W/(m^2\ str)$

#8 Spherical Lambertian Source



a)



incremental ring on hemisphere of Radius R has area

$$2\pi R\sin\theta \cdot R d\theta$$

and subtends (From As) a solid angle of $\frac{2\pi R\sin\theta\, R d\theta}{R^2}$

$$= 2\pi \sin\theta\, d\theta$$

$$\phi = \iint_{drd} dA \int_{1/2\ \text{space}} N\cos\theta\, d\Omega$$

$$= A \int_{1/2\ \text{space}} N\cos\theta(2\pi\sin\theta)d\theta$$

$$= A\, 2\pi \int_0^{\pi/2} N\cos\theta\sin\theta\, d\theta$$

$$| d(\sin^2\theta) = 2\sin\theta\cos\theta\, d\theta$$

$$= A\pi \int_0^{\pi/2} N\, d(\sin^2\theta)$$

$$= A\pi N \sin^2\theta \Big|_0^{\pi/2} = A\pi N$$

$$|\text{Area} = 4\pi r^2$$

$$\phi = 4\pi r^2(\pi N)$$

$$= 4\pi^2 r^2 N \quad [\text{watts}]$$

b) irradiance $= \left[\frac{W}{m^2} \right]$ @ some distance R

$$= \frac{\text{Power}}{\text{Area @ } R} = \frac{4\pi^2 r^2 N}{4\pi R^2} = \pi \left(\frac{r}{R}\right)^2 N$$

c) suppose $R >>> r$ describe an alternative planar Lambertian source

remember from problem (7) The check of part b.

$$E\Big|_{\underline{\text{disk}}} = N\pi \frac{r_1^2}{r_1^2 + R^2}$$

for $R >>> r_1$ $\quad r_1^2 + R^2 \approx R^2$

$$E\Big|_{\underline{\text{disk @ large distance}}} = N\pi \frac{r_1^2}{R}$$