Homework #3

1) FT systems $\qquad F(x) \longrightarrow \boxed{FT} \longrightarrow \mathcal{F}(F(x))$

a) Linear?

IF linear: The output can be written as a summation (integral) of individual responses to unit impulses.

$$\text{-or-} \quad af(x) + bg(x) \longrightarrow \boxed{FT} \longrightarrow a\mathcal{F}(f(x)) + b\mathcal{F}(g(x))$$

True by propeties of fourier Transform.

b)

Can you specify a transfer function for This system? That is: Is The FT both linear & <u>shift invariant</u>?

IF The system is shift invariant Then

$$FT\{f(x-x_0, y-y_0)\} = F(u-x_0, v-y_0)$$

But from The shift theorem we know

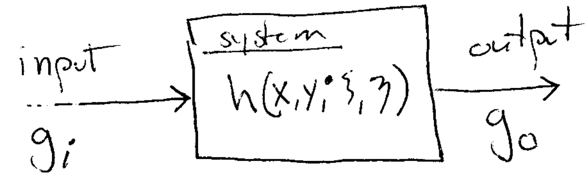
$$FT\{f(x-x_0, y-y_0)\} = F(u,v)e^{-i2\pi(ux_0+vy_0)}$$

$$\therefore \quad F(u-x_0, v-y_0) = F(u,v)e^{-i2\pi(ux_0+vy_0)}$$

So This is a contradiction. because we know a valid Transform exists (eg $\mathrm{sinc}(au)$) That violates This.

$\therefore$ FT is not shift invariant $\Rightarrow \nexists$ transfer function.

#2



$h(x,y;\xi,\eta) \equiv$ The response of the system at point x,y in the output space to a δ function input at (ξ,η) of the input space

The impulse response

If $h(x,y;\xi,\eta)$ is linear the output can be written a a summation (integral) of individual responses to unit impulses

$$g_o(x,y) = \iint_{-\infty}^{\infty} g_i(\xi,\eta)\, h(x,y;\xi,\eta)\, d\xi\, d\eta$$

If the system is shift invariant. the impulse response $h(x,y;\xi,\eta)$ depends only on the distances $(x-\xi)$ and $(y-\eta)$. And we take a simpler notation

$$h(x,y;\xi,\eta) = h(x-\xi,\, y-\eta)$$

Now the output can be written

$$g_o(x,y) = \iint_{-\infty}^{\infty} g_i(\xi,\eta)\, h(x-\xi,\, y-\eta)\, d\xi\, d\eta$$

which is the two dimensional convolution, written in a simpler notation

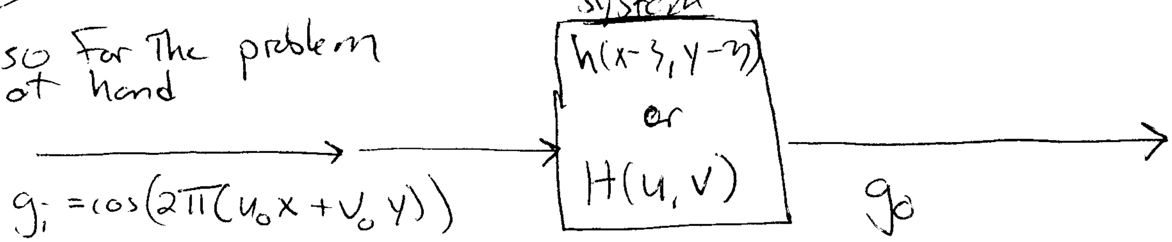
$$g_o(x,y) = g_i(x,y) \otimes h(x,y)$$

The reason we like linear, shift-invariant systems is now revealed when we take the Fourier Transform of the input/output relation:

$$\mathcal{FT}\left(g_o(x,y) = g_i(x,y) \otimes h(x,y) \right)$$

$$\Rightarrow \; G_o(u,v) = G_i(u,v)\, H(u,v)$$

so for the problem at hand



$$G_o(u,v) = G_i(u,v)\, H(u,v)$$

$$G_i(u,v) = \iint \cos(2\pi(u_o x + v_o y))\, e^{-i2\pi(ux+vy)}\, dx\,dy$$

$$= \iint \tfrac{1}{2}\left[e^{i2\pi(u_o x + v_o y)} + e^{-i2\pi(u_o x + v_o y)} \right] e^{-i2\pi(ux+vy)}\, dx\,dy$$

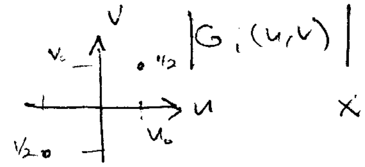
$$= \iint \tfrac{1}{2} e^{i(2\pi u_o x + 2\pi v_o y - 2\pi u x - 2\pi v y)}\, dx\,dy + \tfrac{1}{2}\iint e^{-i(2\pi u_o x + 2\pi v_o y + 2\pi u x + 2\pi v y)}\, dx\,dy$$

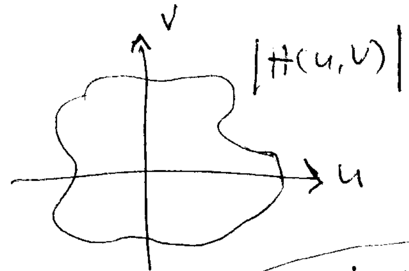
$$= \iint \tfrac{1}{2} e^{-i2\pi[(u-u_o)x + (v-v_o)y]}\, dx\,dy + \iint \tfrac{1}{2} e^{-i2\pi[(u+u_o)x + (v+v_o)y]}\, dx\,dy$$

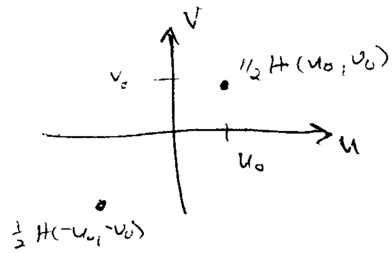
$$= \tfrac{1}{2}\delta(u-u_o, v-v_o) + \tfrac{1}{2}\delta(u+u_o, v+v_o)$$

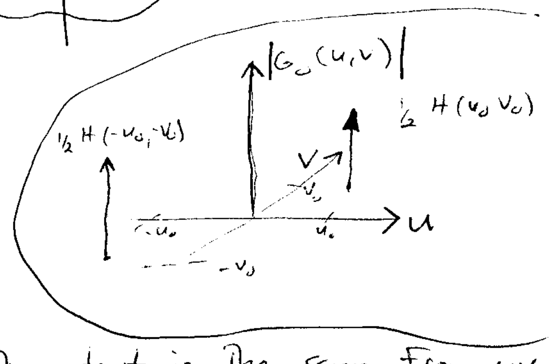
so

$$G_o(u,v) = G_i(u,v) \times H(u,v)$$

$=$ 

$\times$ 

$=$ 

$\Longrightarrow$ 

$\Longrightarrow$ clearly the output is the same frequency as input. with a different Amplitude & phase

$$G_o(u,v) = \frac{1}{2} H(u_o, v_o)\, \delta(u-u_o, v-v_o) + \frac{1}{2} H(-u_o, -v_o)\, \delta(u+u_o, v+v_o)$$

$$g_o(x,y) = \mathcal{F}^{-1}(G_o(u,v))$$

$$= \frac{1}{2} H(u_o, v_o)\, e^{i2\pi(u_o x + v_o y)} + \frac{1}{2} H(-u_o, -v_o)\, e^{-i2\pi(u_o x + v_o y)}$$

$$\boxed{g_o(x,y) = |M| \cos[2\pi(u_o x + v_o y) - \phi]}$$

w/

$$|M| = 2\sqrt{\frac{1}{2}\frac{1}{2} H(u_o, v_o)\, H(-u_o, -v_o)}$$

$$\phi = \tan^{-1}\left(\frac{i\left[\frac{1}{2} H(-u_o, -v_o) + \frac{1}{2} H(+u_o, +v_o)\right]}{\left[\frac{1}{2} H(-u_o, -v_o) - \frac{1}{2} H(u_o, v_o)\right]}\right)$$

see details on next page.

Details

$$\Rightarrow Ae^{i\theta x} + Be^{-i\theta x} = D\sin(\theta x) + E\cos(\theta x)$$

$$= D\left(\frac{e^{i\theta x} - e^{-i\theta x}}{2i}\right) + E\left(\frac{e^{i\theta x} + e^{-i\theta x}}{2}\right)$$

$$= \left(\frac{D}{2i} + \frac{E}{2}\right)e^{i\theta x} + \left(\frac{-D}{2i} + \frac{E}{2}\right)e^{-i\theta x}$$

$\therefore$

$$D + iE = 2iA \Rightarrow -2iA + iE = -D$$

$$-D + iE = 2iB \Rightarrow -2iA + iE + iE = 2iB$$

$$-2iA + 2iE = 2iB$$

$$\boxed{E = B + A}$$

$$D = 2iA - iE = 2iA - i(B+A)$$

$$= 2iA - iB - iA$$

$$= iA - iB$$

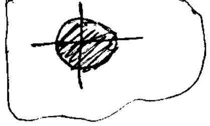
$$\boxed{D = i(A - B)}$$

Now

$$D\sin(\theta x) + E\cos(\theta x) = |M|\cos(\theta x - \phi)$$

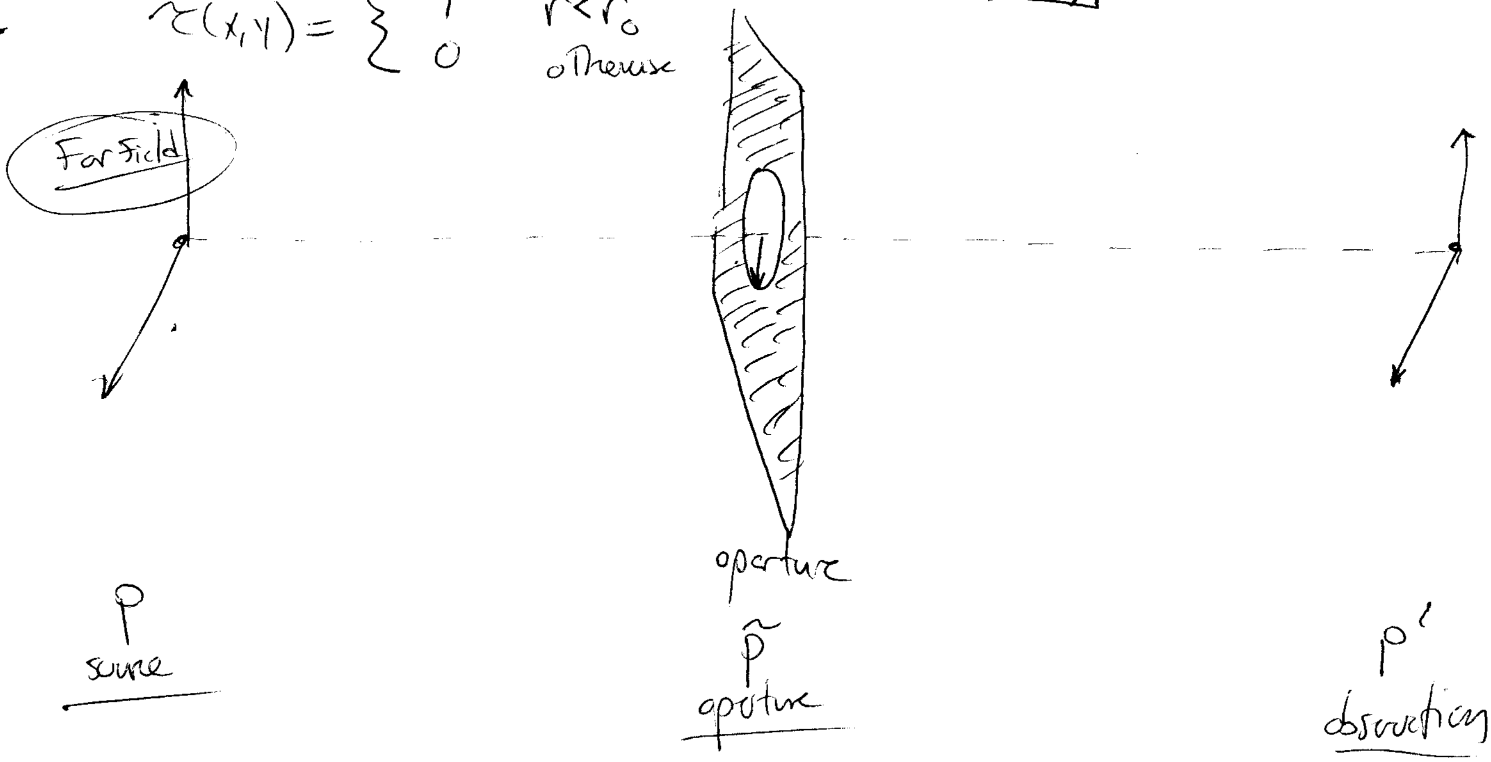
$$|M| = \sqrt{D^2 + E^2} = \sqrt{i^2(A-B)^2 + (B+A)^2} = 2\sqrt{AB}$$

$$\phi = \tan^{-1}\left(\frac{E}{D}\right) = \tan^{-1}\left(\frac{B+A}{-i(B-A)}\right) = \tan^{-1}\left(\frac{i(B+A)}{B-A}\right)$$

#3 a) $I(u,v)$ for 

First lets Find $T(u,v)$ For 

$$\tau(x,y) = \begin{cases} 1 & r<r_0 \\ 0 & \text{otherwise} \end{cases}$$



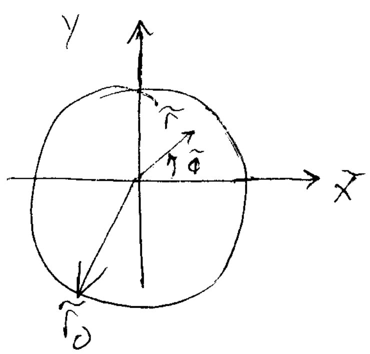
Far Field approximation

where distance as assumed to be linear in $\tilde{x}$ & $\tilde{y}$

(i.e $\sqrt{1+x} \approx 1 + \frac{1}{2}x$

we find that the observed Field at P' due to a point source at P w/ an aperture τ is given by

$$E'(u,v) \propto T(u,v) = \iint_{-\infty}^{\infty} \tau(\tilde{x},\tilde{y}) e^{-i2\pi(u\tilde{x} + v\tilde{y})} d\tilde{x}\, d\tilde{y}$$



$\tilde{x} = \hat{r}\cos\tilde{\phi}$

$\tilde{y} = \hat{r}\sin\hat{\phi}$

$$T(u,v) = \iint_{-\infty}^{\infty} \begin{pmatrix} 1 & \text{inside circle} \\ 0 & \text{outside circle} \end{pmatrix} e^{-i2\pi(u\hat{r}\cos\hat{\phi} + v\hat{r}\sin\hat{\phi})} \, \hat{r}\, d\hat{r}\, d\tilde{\phi}$$

Further exploit circular symmetry

let $\rho \equiv \sqrt{u^2+v^2}$

$u = \rho\cos\phi'$

$v = \rho\sin\phi'$

$$T(\rho,\phi') = \int_0^{r_0}\int_0^{2\pi} 1 \cdot e^{-i2\pi(\rho\hat{r}\cos\hat{\phi}\cos\phi' + \rho\hat{r}\sin\hat{\phi}\sin\phi')} \, \hat{r}\, d\hat{r}\, d\hat{\phi}$$

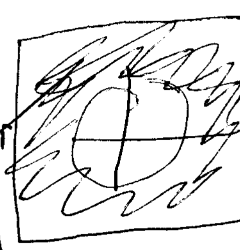
$$= \int_0^{r_0}\int_0^{2\pi} e^{-i2\pi(\rho\tilde{r})(\cos\tilde{\phi}-\phi')} \, \tilde{r}\, d\hat{r}\, d\phi$$

note: (eg) 6.45 $\quad J_0(w) = \frac{1}{2\pi}\int_{\emptyset}^{2\pi} \exp[-iw\cos(\tilde{\phi}-\phi')]\, d\tilde{\phi}$

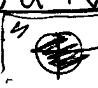
$\int_0^x \xi J_0(\xi)\, d\xi = x J_1(x)$

$$T(\rho,\phi') = 2\pi\int_0^{r_0} \tilde{r}\, J_0(2\pi\rho\tilde{r})\, d\tilde{r} = \frac{r_0}{\rho} J_1(2\pi\rho r_0)$$

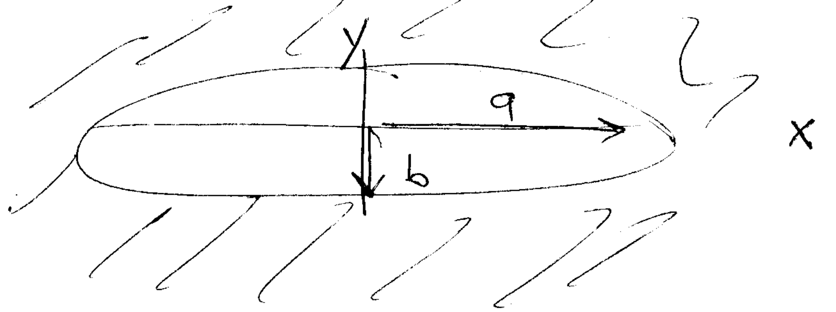
$$T(u,v) = \frac{r_0}{\sqrt{u^2+v^2}} J_1\left(2\pi\sqrt{u^2+v^2}\, r_0\right)$$ This is for



$$\boxed{T(u,v) \text{ of } \left[\text{"}\right. \,\text{〈sketch〉}\, \left.\right] = \delta(u,v) - \frac{r_0}{\sqrt{u^2+v^2}} J_1\left(2\pi\sqrt{u^2+v^2}\, r_0\right)}$$



b)



$$f(x,y) = \begin{cases} 1 & \text{inside ellipse} \\ 0 & \text{out} \end{cases} \Rightarrow F(u,v)?$$

equation for ellipse $r^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$x = ar\cos\theta \qquad y = br\sin\theta$

$$F(u,v) = \iint_{\text{ellipse}} e^{-i2\pi(ux+vy)}\, dx\, dy$$

change of variables

Note in general: (see attached copies from Hildebrand)

$$\iint_D w(x,y)\,dx\,dy = \iint_{D^*} w(u_1,u_2) \underbrace{\left|\frac{\partial(x,y)}{\partial(u_1,u_2)}\right|}_{\text{Jacobian}} du_1\,du_2$$

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

so

$$F(u,v) = \int_0^{2\pi} d\theta \int_0^1 abr\,dr\, e^{-i2\pi(uar\cos\theta + vbr\sin\theta)}$$

let $\begin{cases} ua = \rho\cos\phi \\ vb = \rho\sin\phi \end{cases}$ $\quad \rho^2 = u^2a^2 + v^2b^2$

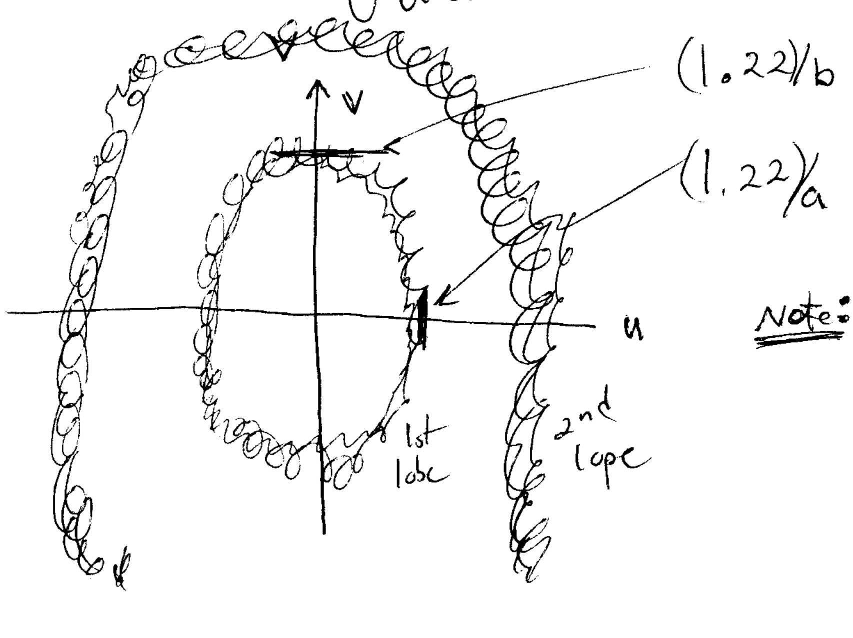
$$F(\rho,\phi) = \int_0^{2\pi}\int_0^1 abr\, e^{-i2\pi(r\rho\cos\theta\cos\phi + r\rho\sin\theta\sin\phi)}\, dr\, d\phi$$

$$= ab\int_0^{2\pi}\int_0^1 r\, e^{-i2\pi\rho r\cos(\theta-\phi)}\, dr\, d\phi$$

$$= 2\pi\, ab\int_0^1 r\, J_0(2\pi\rho r)\, dr$$

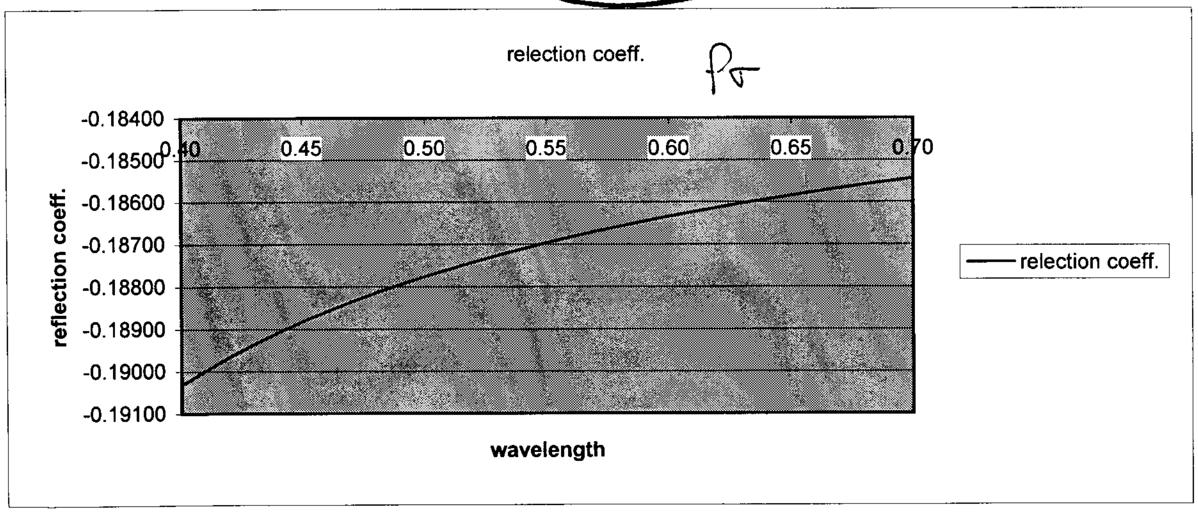
$$= ab\,\frac{J_1(2\pi\rho)}{\rho}$$

$$F(u,v) = ab\,\frac{J_1\left(2\pi\sqrt{u^2a^2+v^2b^2}\right)}{\sqrt{u^2a^2+v^2b^2}}$$



Note: the First lobe of the FT is also an ellipse oriented 90° wrt the original ellipse

wavelength	n	relection coeff.	[redacted] Energy Refl. Coeffi $= \lvert\rho\rvert^2$
0.40	1.47011	-0.19032	0.0362
0.42	1.46809	-0.18966	0.0360
0.44	1.46634	-0.18908	0.0358
0.46	1.46483	-0.18859	0.0356
0.48	1.46350	-0.18815	0.0354
0.50	1.46232	-0.18776	0.0353
0.52	1.46128	-0.18741	0.0351
0.54	1.46034	-0.18710	0.0350
0.56	1.45949	-0.18682	0.0349
0.58	1.45873	-0.18657	0.0348
0.60	1.45803	-0.18634	0.0347
0.62	1.45739	-0.18613	0.0346
0.64	1.45681	-0.18594	0.0346
0.66	1.45626	-0.18575	0.0345
0.68	1.45576	-0.18559	0.0344
0.70	1.45529	-0.18543	0.0344
	average:	-0.18729	0.0351



5a) Field Reflection

Energy Reflection

$$\rho_\sigma = \frac{1 - n(\lambda)}{1 + n(\lambda)} \quad \text{-or-} \quad R = \rho_\sigma^2$$

both accepted

5b) a couple of things were accepted.

$\langle \rho_\sigma \rangle \equiv$ average of $\rho_\sigma \cong -.187$

or

$\langle R \rangle \equiv$ average power reflected $\approx 3.5\%$

i.e. what would actually be observed w/ your eye or a sensor

This is actually what was wanted in the question.

flux density $\equiv$ flux per unit area at a point on a surface $[W/m^2]$

wavelength	n	relection coeff.	Power Refl. Coeffi
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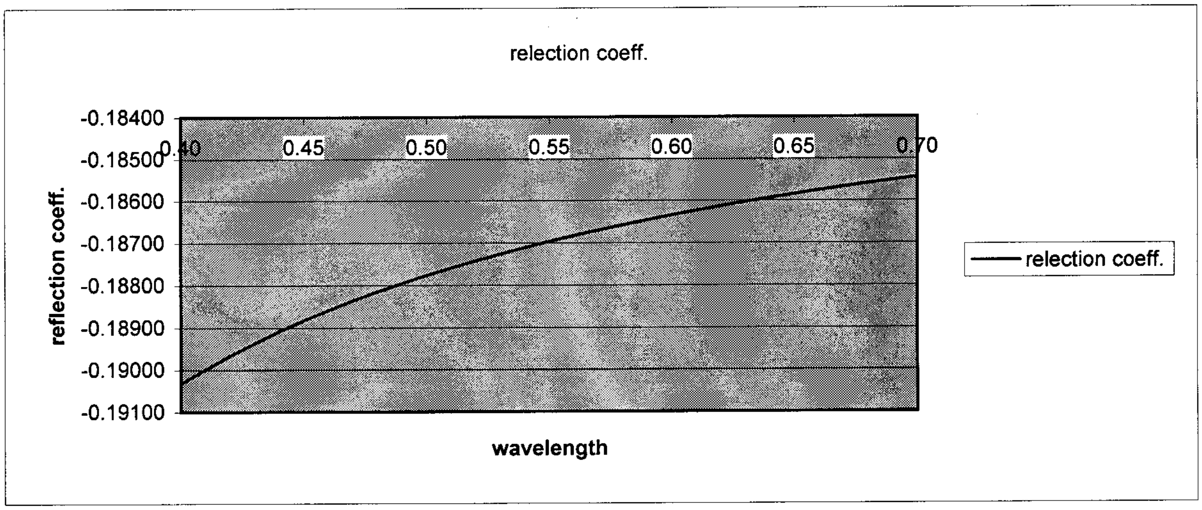
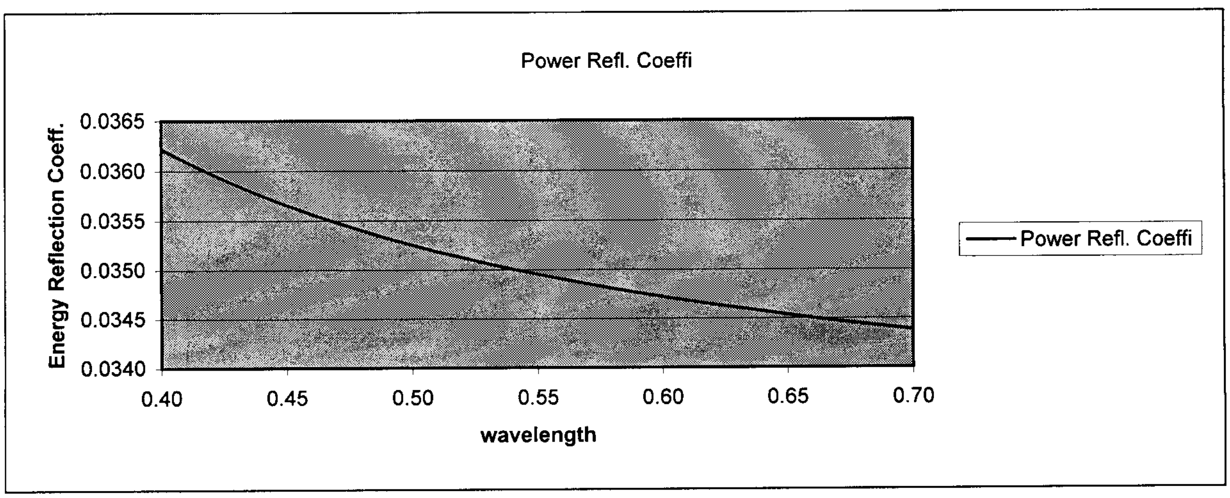
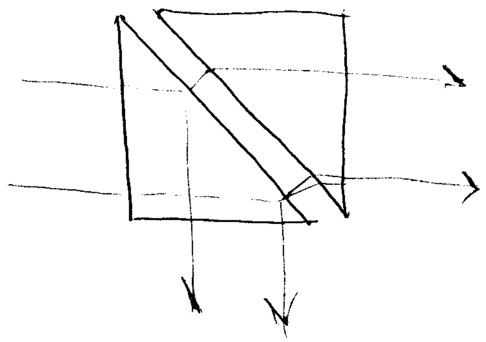




#6



$n = 1.6$

$\lambda = 632.8\ nm$ (π polarized)

find size air gap that produces a Transmission coefficient of 50%

energy Transmission

- $$T_{\pi} = 2e^{-2d/\delta}(1 - \cos 2\phi_{\pi})$$ eq 2.84

$d \equiv$ gap $\qquad \phi_{\pi}$ - π phase angle

$\delta \equiv$ skin depth

$$d = -\frac{\delta}{2} \ln\left[\frac{T_{\pi}}{2(1-\cos 2\phi_{\pi})}\right]$$

- δ skin depth From eq 2.82

$$\delta = \frac{\lambda_0}{2\pi\sqrt{n^2 \sin^2\theta - n'^2}}$$

- $$\tan\frac{\phi_{\pi}}{2} = \frac{n^2}{n'^2}\gamma \Rightarrow \phi_{\pi} = 2\tan^{-1}\left(\frac{n^2}{n'^2}\gamma\right)$$ (follows from 2.83)

γ - describes evanescent wave (analog to wave vector that decays)

$$\gamma = \frac{\sqrt{n^2 \sin^2\theta - n'^2}}{n\cos\theta}$$

so all together

$$d = -\frac{\delta}{2} \ln\left[\frac{T_{\pi}}{2(1-\cos 2\phi_{\pi})}\right]$$

w/

$$\delta = \frac{\lambda_0}{2\pi(n^2\sin^2\theta - n'^2)^{1/2}}$$

$$\phi_{\pi} = 2\tan^{-1}\left(\frac{n^2}{n'^2}\gamma\right)$$

$$\gamma = \frac{\sqrt{n^2\sin^2\theta - n'^2}}{n\cos\theta}$$

with

$\theta = \frac{\pi}{4}$ $\quad T_{\pi} = .50$ $\quad n' = 1$ $\quad n = 1.6$

$\lambda = 638.8\,nm$

//

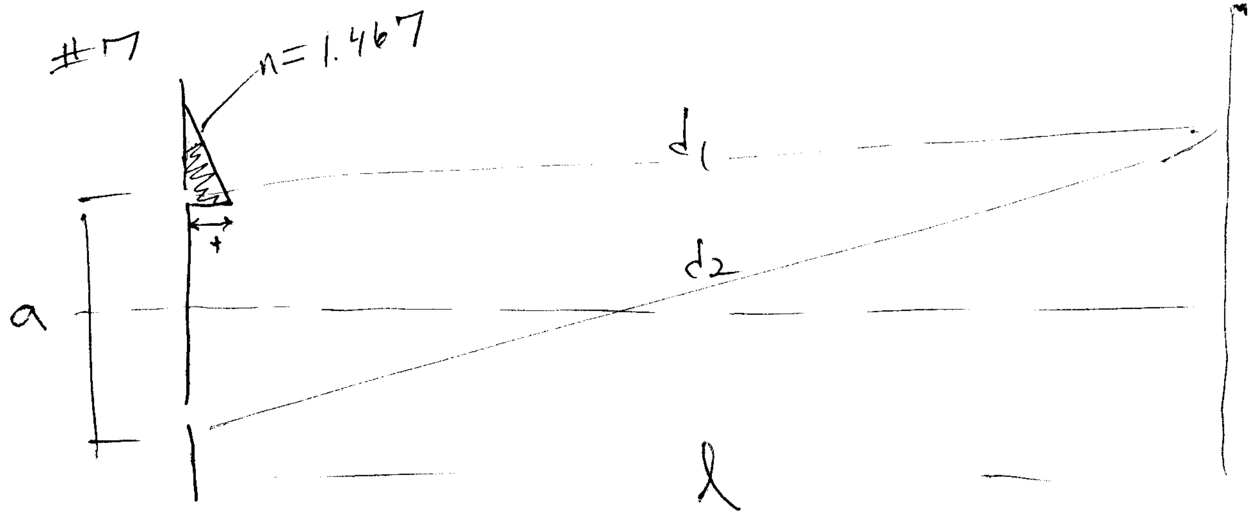
$\gamma = .4677$

$\delta = 190.3301\ nm$

$\phi_{\pi} = 1.7499\ rad$

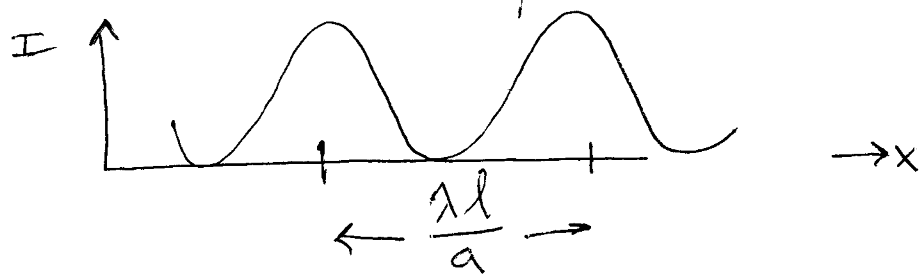
$$\boxed{d = 194.82\ nm}$$

#7



- central max shifts by 3.4 fringes to one side
- Find max Thickness of wedge

we know that for 2 point interference



so we have a <u>distance shift</u> of $3.4\left(\frac{\lambda l}{a}\right)$

at maximum the phase difference between two paths is zero.

phase difference:

$$\phi = \frac{2\pi}{\lambda}(OPL_1 - OPL_2)$$

$$= \frac{2\pi}{\lambda}\left[(nt + d_1 - t) - d_2\right]$$

$$= \frac{2\pi}{\lambda}\left[(n-1)t + d_1 - d_2\right]$$

$$d_1^2 = \ell^2 + \left(\frac{a}{2} - x\right)^2$$

$$d_1 = \ell\sqrt{1+\left(\frac{a/2 - x}{\ell}\right)^2} \simeq \ell\left[1 + \frac{1}{2}\left(\frac{a/2 - x}{\ell}\right)^2\right]$$

$$d_2 = \qquad \simeq \ell\left[1 + \tfrac{1}{2}\left(\frac{a/2 + x}{\ell}\right)^2\right]$$

$$\phi = \frac{2\pi}{\lambda}\left[(n-1)t + \overbrace{\frac{ax}{\ell}}^{d_1 - d_2}\right]$$

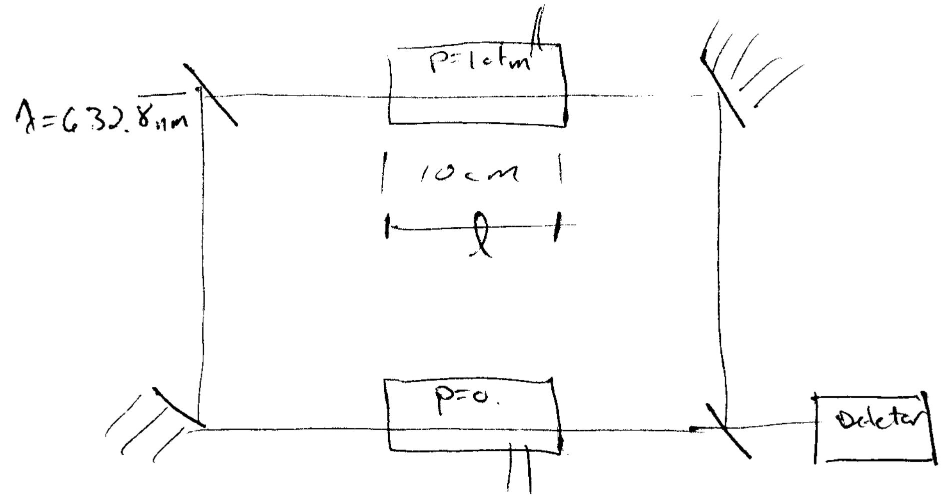
so

at $x = 3.4\,\frac{\lambda\ell}{a}$; $\phi = 0$

$$\phi = \frac{2\pi}{\lambda}\left[(n-1)t + \frac{a}{\cancel{\ell}}(3.4)\frac{\lambda\cancel{\ell}}{\cancel{a}}\right] = 0$$

$$\boxed{t = \frac{3.4\lambda}{(n-1)} = \frac{3.4\lambda}{.467} = 7.2805\ \lambda}$$

\# 8 KF 5.37



100 minima are observed as cell is evacuated. ← goes through 2π 100 times

n for gas is $n = 1 + Ap$ — pressure

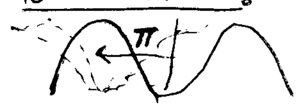
$$\phi = \frac{2\pi}{\lambda_0}(OPD) \quad \leftarrow \text{optical path difference}$$

$$= \frac{2\pi}{\lambda_0}(n-1)\ell$$

$$\phi = \frac{2\pi}{\lambda_0}\left[(1+Ap)-1\right]\ell$$

$$\frac{\phi \lambda_0}{2\pi \ell}\frac{1}{p} = A$$

to be strict:



The first minimum occurs after π shift, then every 2π after that

so actually ϕ is

$200\pi - \pi$
$= 199\pi$

$$\boxed{A = \frac{2\pi \cdot 100 \cdot \lambda_0}{2\pi \;\ell\; p} = 6.328 \times 10^{-4}\ \frac{1}{atm}}$$

if strict:

$$\boxed{A = \frac{199\pi \cdot \lambda_0}{2\pi\,\ell\,p} = 6.296 \times 10^{-4}\ \frac{1}{atm}}$$

accepted either answer