Geometric and Photometric Constraints: Motion and Structure from Three Views

by

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Abstract

This thesis addresses the problem of recovering both the camera motion and scene structure given a sequence of images obtained from a camera moving through a rigid environment. The core of the thesis is the *model-based brightness constraint*, a new method for computing the camera motion and dense scene structure from three views. We combine photometric constraints together with geometric constraints to directly compute the camera motion from the image brightness derivatives (spatial and temporal). This type of method is dubbed a *direct method* because it does not require prior computation of optical flow or point correspondences. We describe the advantages and limitations of this method and present results with real images.

We apply the same framework to the analysis of image sequences obtained from a moving stereo rig. We show that the motion of the stereo rig can be computed directly from the image derivatives using linear equations. Correspondences are not required. The motion estimates between stereo pairs enable us to combine depth maps from all the pairs in the sequence to form an extended scene reconstruction and we show results from a real image sequence.

The tensor brightness constraint is intimately connected to 3D reconstruction from line correspondences. For line correspondences, the linear solution is degenerate when all the lines come from a Linear Line Complex (i.e. when all the lines intersect a single line in space). This configuration is quite common in practice and in particular, it arises in the context of the model-based brightness constraint. We show a method for overcoming the degeneracy of the general case and show some experimental results.

To accurately disambiguate between camera translation and rotation a wide field-of-view is required. For wide angle lenses, non-linear lens distortion can be a significant factor and affect the accuracy of motion and structure estimates. We present a new method for determining the lens distortion parameters using geometric constraints on a set of point correspondences in multiple views. There is no need to know the 3D location of the points.

Thesis Co-Supervisor: Prof. W. Eric L. Grimson

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Chapter 1

Introduction

Motion gives us a strong perception of the three dimensional world. As we walk down the street or through the woods we sense the space around us. We sense that we are moving. We know if the path ahead is clear. We know when to duck to avoid hitting a low overhead branch. We might notice an opening in the wall to our left. How do we explain this?

How can we get a robot to do the same? More precisely, given a sequence of images obtained from a camera moving through a rigid environment, how do we recover both the camera motion and scene structure? That problem is the focus of this thesis.

The geometry of multiple views of a 3D scene is well understood. There exist geometric constraints, which relate corresponding points and lines in multiple views to the camera geometry. These constraints take the form of the trilinear tensor equations for three views, and the epipolar constraints for two views. Given a set of feature correspondences we can recover the camera geometry, and in particular camera motion (or the relative position of multiple cameras). But finding correspondences is a hard problem, and the features must be recovered accurately in order to correctly recover camera motion.

Feature correspondence, whether optical flow or discrete features, is based in some form or another, on the constant brightness assumption. That is, we assume the same feature in two views share some photometric properties. In it's strictest form, the *constant brightness constraint*, assumes that the brightness of the corresponding point does not change between views. Alternatively, we might look at some function of the brightness, possibly a nonlinear function such as brightness edges. The constant brightness constraint is a good approximation for many surfaces in the real world, especially if the motion is small. It is well known that the constant brightness constraint is not strong enough to give us true correspondences. Local measurements, for example, cannot give 'optical flow' but only *normal flow*, that is, the image flow estimates in the direction of the image brightness gradient. This is a particular problem in scenes with long, nearly straight, edges. We will show some examples later in the thesis.

In this thesis we have developed the *model-based brightness constraints*, where we combine geometric motion models with the brightness constraint. This provides us with a stronger constraint that can be used for direct estimation of structure and motion. We can threreby bypass the correspondence problem, and recover the camera motion directly from the image brightness values. This results in accurate

motion estimates, and good 3D reconstruction, in very challenging scenes.

What happens if we know some but not all of the camera geometry? How can this be used in the geometric constraints? We have studied these questions in the context of stereo image sequences, where we know the displacement between the two cameras, but do not know the motion of the stereo rig as a whole. In this case multi-view geometric constraints provide an elegant and simple way to combine both motion and stereo. This is true if we have feature correspondences and can also be used for direct methods. The latter can be implemented using the framework of model-based constraints. We show as an example, how we can easily recover the motion of the stereo rig directly from image brightness values. One can then use the images from both pairs combined to compute a dense depth map at every camera location. Using the motion estimates between stereo pairs we bring all the depth reconstructions into one coordinate frame.

The model-based brightness constraints are intimately connected to 3D reconstruction from line correspondences. We prove that for line correspondences, the linear solution is degenerate when all the lines come from a Linear Line Complex (i.e. when all the lines intersect a single line in space). This configuration is shown to be quite common in practice and in particular, it arises in the context of the model-based brightness constraints. We show a method for overcoming the degeneracy of the general case and show some experimental results.

In order to correctly disambiguate between rotation and translation, we need to use a wide field-of-view. This requires wide angle lenses, which typically have large amounts of non-linear lens distortion. We investigated how lens distortion affects the geometric constraints. In the process, we developed a new method for lens distortion calibration using a set of point correspondences in multiple views. A calibration object is not required, and there is no need to know the 3D location of the points. This makes the method very practical to use.

Where Versus What

In this thesis we focus on determining *where* objects are in the scene relative to the camera and how the camera is moving relative to the scene. This is what we call the perception of 'space'. To accurately recover camera motion and distance to an object requires a perspective camera model and wide-angle view. (In Section 3.7.5 we show that a wide field of view is required to disambiguate camera rotation and translation).

The other part of the structure from motion problem is the perception of the *shape* of an object. We can think of this as answering the question *what* is the object. The main requirements are reasonable image resolution, reasonable camera baseline and camera geometry that disambiguates rotation and translation. While not the main focus of this thesis, we show some good shape reconstruction results (for example Figure 1-2). Good results have also been achieved using orthographic or weak perspective camera models and an object centered coordinate system [79, 35, 81, 75, 54]. Using the orthographic camera model, the distance to the object cannot be recovered.

The rest of the chapter

In the following sections we will briefly expand on the four main topics of the thesis: the model-based brightness constraints, combining motion and stereo, the linear degeneracy of the linear line complex, and finally, lens distortion calibration. In Section 1.5 we provide an example application: extended scene reconstruction from a stereo image sequence. In Section 1.6 we describe the organization of the rest of the thesis.

1.1 Model-based Brightness Constraints: Motion and Structure from Three Views

The core of the thesis is a new method for computing the camera motion and dense scene structure from three views in an image sequence. This is done without first computing point correspondences or optical flow. Instead we combine the photometric constraints between pairs of views together with the geometric constraints between three views to directly compute the camera motion. After computing the motion we can compute dense structure, in other words, the depth to every pixel in the image. This type of method is dubbed a *direct method* because it does not require prior computation of optical flow.

This is in contrast to previous approaches which in general first find point correspondences between consecutive images using photometric (i.e. brightness) constraints. After computing the correspondences the camera motion between two or three views and the 3D structure of the scene are recovered using geometric constraints. Here we include both 'optical flow' based methods [42, 43, 28, 55, 77] and feature based methods [79, 16, 65, 67, 4]. The main problem with these approaches is that computing the correspondence is in itself a difficult and error prone process. Dense correspondence methods such as 'optical flow' suffer from the aperture problem and it is necessary to assume smoothness in the flow [33, 43, 49]. Sparse methods try to find matching features such as lines or corner points. It is still an open subject of research as to what makes a good feature[11, 83, 16]. Some points and lines in the image do not correspond to any real feature in 3D (e.g. the occluding contour of a cylinder or of the head in Figure 1-1). This issue is discussed in more depth in the introduction to Chapter 3. These errors in computing the correspondences get carried through to the motion and structure estimation stage.

Our method avoids these problems by directly computing the motion from image brightness measurements without first computing correspondences, and by using information from all the pixels in the image to compute the motion estimates. The wide area of support for the computation means that the motion estimates are less affected by regions where the brightness constraints are violated, such as on occluding contours.

We now provide the key ideas behind the new method. The *optical flow constraint equation* [33] provides a matching constraint between each point in one image, and a line passing through the corresponding point in the second image. In other words,

for every point in one image it gives us the equation of a line along which the corresponding point must lie in the second image. By *point* we refer to every pixel in the image which has a none zero brightness derivative. The equation is given in terms of the spatio-temporal derivatives of the image brightness, and the line is parallel to the iso-brightness contour in the first image. The 'optical flow constraint equation' is a first order approximation and assumes that the image motion (and hence typically, the camera motion) is very small.

There are no geometric constraints on point-line correspondences between two views, as can be seen using the following reasoning. A point in the image defines a line in space. A line in the second image defines a plane in space. A line and a plane always intersect. Therefore, given a set of points in one image and a set of corresponding lines in the second image, for every camera geometry there exists a valid 3D interpretation and no constraint on the geometry exists. We must therefore use three view geometry, where a point in one view, and lines through the corresponding points in two other views, provide a constraint, which can be written in the form of the *trilinear tensor* [57, 27, 69]. The 27 coefficients of the 'trilinear tensor' encapsulate the camera motions and the internal parameters of the camera such as the focal length. The new method can therefore be viewed as a necessary extension of the 'direct methods' of Horn & Weldon [34] from two views (one motion) to three views (two motions).

By combining the optical flow constraint equation with the geometric model of the 'trilinear tensor' we obtain the *tensor brightness constraint* [74] that describes the relationship between the spatio-temporal brightness derivatives at each pixel in the image, with the camera parameters modeled by the 27 coefficients of the trilinear tensor. This 'tensor brightness constraint' provides one linear equation per pixel in the image which results in a highly over-constrained set of equations. The 'tensor brightness constraint' is valid for the most general case (projective) where the cameras undergo general motion and we do not know the internal camera parameters which might vary from frame to frame.

We then proceed through a hierarchy of simpler motion models first by assuming calibrated cameras and small rotations, then by assuming the Longuet-Higgins & Prazdny small motion model [42], where we assume also that the camera translation in depth is small relative to the depth of the objects in the scene. Thus we have a hierarchy of *model-based brightness constraints*. In particular the Longuet-Higgins & Prazdny model leads to a 15-parameter brightness constraint, where the 15 parameters are bilinear combinations of the unknown translations and rotations. One constraint equation for each pixel leads to a set of linear homogeneous equations in the 15 intermediate parameters.

By avoiding the need to explicitly detect feature points we can use information from areas where gradient information is weak such as shadow edges. This advantage is highlighted in Figure 1-1. The images are of low contrast without many distinctive features. The strongest 'features' are the occluding contours of the head which do not correspond to a real feature in space (as the camera moves different points on the head form the occluding contour). Nevertheless, our method produces accurate motion and depth estimates (Figure 1-2). The full quantitative and qualitative results

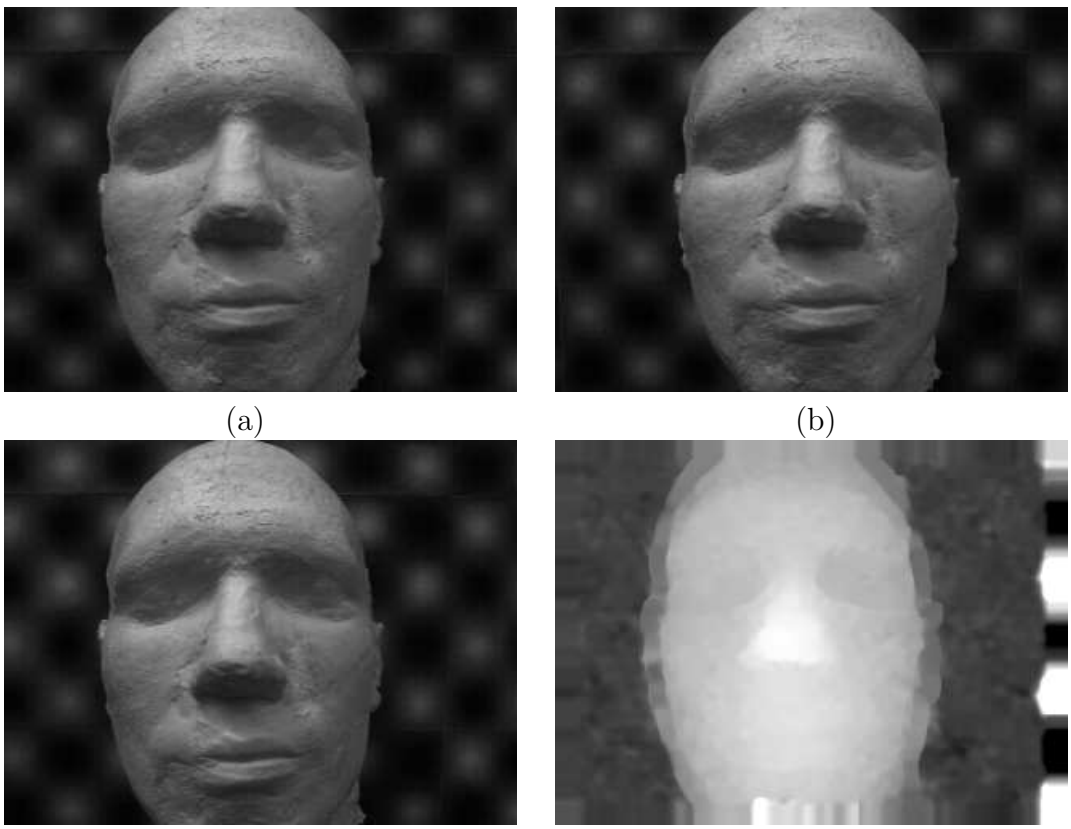


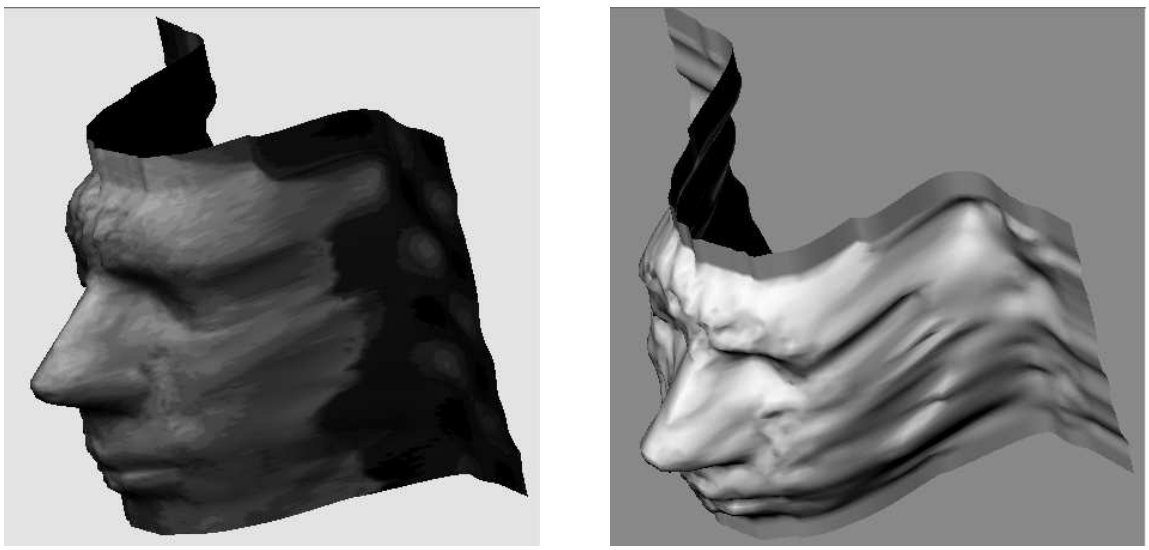
Figure 1-1: *The three input images (a,b,c) used for 3D reconstruction. Motion* $(a \rightarrow b)$ *was a horizontal translation with a* 1.8^o *rotation around the* Y *axis. Motion* $(a \rightarrow c)$ *was a vertical translation. This is a challenging set of images because the the texture is smooth with low contrast. The strongest edge 'features' appear along the occluding contour of the head but this edge does not in fact correspond to a real feature in the world. (d) The estimated depth map. The estimated rotation was* 1.62^o*. There are of course errors around the boundary of the image where there is no overlap between the images.*

are described in Chapter 4.

1.2 Combining Motion and Stereo

Stereo and motion together form a powerful combination [23, 91] with a variety of applications from camera motion estimation to extended scene reconstruction. A stereo system (with two or more cameras) can give accurate depth estimates but only from one view point. To build a complete model of a scene we would like to combine together multiple 3D views. One solution is to use a moving stereo rig (two cameras mounted onto the same rigid frame).

The standard approach to reconstruction from a moving stereo rig [39, 91, 85, 16] has been to first compute a depth map from each pair (or compute the depth of a



(a) (b)

Figure 1-2: *3D rendering of the depth map in Figure 1-1d. (b) texture map removed to show detail.*

set of features points), together with a confidence measure for each depth estimate. Next, one registers the stereo pair with the current 3D model to compute the motion, and finally, one updates the model (using a Kalman filter or a batch method). In contrast, Navab et al [51] combine motion and stereo for the reconstruction of motion and structure from line correspondences. Given two distinct non-parallel lines in two stereo image pairs they obtain a set of 8 linear equations in the 6 motion parameters. The method requires accurate tracking of straight lines in an image.

In Chapter 5 we describe a new method for directly estimating the motion of a stereo rig . We show that given two stereo pairs one can compute the motion of the stereo rig directly from the image derivatives (spatial and temporal). Correspondences are not required.

Directly computing the motion of the stereo rig simplifies the problem of registering all the depth reconstructions into one coordinate frame. Our method gives good estimates of the camera motion between frames. There is no need to explicitly compute confidence values for the depth estimates, since the causes for error are taken into account implicitly in the equations. By concatenating these motion estimates, one can bring all the depth maps into one coordinate frame.

The core of the method is to apply the multi-view geometric constraints (Chapter 2) to a situation where some, but not all, of the camera motions are known. We first develope the equations for feature correspondences in the case of calibarted cameras. In order to avoid the need to find the correspondences we use then use the 15-parameter model-based brightness constraints (Chapter 3). We apply the model-based brightness constraint to the case where the motion between one pair of views is known. In general, the model-based brightness constraint equations are bilinear in the motion parameters but if one of the motions is known they result in a set of linear equations for the second motion parameters. There is one equation for each point in

the image and 6 unknowns (the translation and rotation parameters), which results in a highly over-constrained, and very stable, computation.

The use of a stereo pair rather than a single image sequence (monocular structure from motion) has further advantages. The method is stable in the case of collinear motion, in the case of pure rotation and also for planar objects. There is no scale ambiguity since all motions are relative to the known baseline length [23]. We present an example of a 3D reconstruction from a sequence of 50 stereo image pairs.

1.3 Degeneracy of the Linear Solution to the Trilinear Tensor in the Case of a Linear Line Complex

The tensor brightness constraint combines both photometric and geometric constraints. The result is a system of linear homogeneous equations (one equation per pixel) in the 27 unknown coefficients of the trilinear tensor. A unique linear solution would be abtainable if the rank of the linear system was 26 but we find that the rank is 23. Is this a purely geometric phenomenom or is it because of the photmetric constraints? We show in Chapter 6 that the degeneracy is geometric in origin and that it is connected to reconstruction from line correspondences.

A set of line correspondences among three views gives rise to a linear set of equations where the unknowns are the camera and motion parameters represented by the $3 \times 3 \times 3$ trilinear tensor. The image measurements are the coordinates of corresponding lines. The 27 coefficients of the trilinear tensor are not independent and there exist 9 non-linear constraints that reduce the tensor's degrees of freedom to 18.

In general this system of equations has a unique linear solution but for particular 3D line configurations in space this system of equations is degenerate. There are some *critical line configurations* from which a unique solution is not possible (see [44, 9, 50]). There are other configurations where a linear solution is degenerate but the additional non-linear constraints lead to a unique solution. When the sample of features is taken from a configuration of lines that have a common intersection, known as a *Linear Line Complex* (LLC), then the rank of the linear system reduces from 26 (in the general case) to 23 — yet, there exists a *unique* solution for the tensor when the non-linear admissibility constraints are accounted for.

The line configuration described by an LLC, rather than being some obscure case, is in fact quite typical. It arises, as a particular example, in the case of the tensor brightness constraint. In also include many urban scenes such as in Figure 1-3. In Figure 1-3a the common intersecting line is the edge of the building. All horizontal lines on the two faces of the building meet the edge in the image plane, and the vertical lines meet the edge at infinity. Note also that the vertical line representing the lamp-post also meets the edge of the building (at infinity) and is thereby included in the LLC configuration. This leaves very few lines (the sidewalk and the oblique line of the lamp-post) not part of the LLC. The common line in Figure 1-3b is the edge of the book-case leaning on the wall. All vertical lines in the scene intersect that

(a) (b)

Figure 1-3: *Typical urban indoor and outdoor scenes. Most of the 3D lines in each scene intersect a single line in space and therefore form a Linear Line Complex. (a) The all lines on the two faces of the building and the vertical lines of the lamp post intersect the edge (corner) of the building (possibly at infinity). (b) All vertical lines in the scene and all the horizontal lines on the bookcase and file cabinet intersect the edge of the bookcase leaning on the wall. Linear methods for 3D reconstruction from lines will fail for these scenes although unique solutions exist using non-linear methods.*

line at infinity and the horizontal lines of the file cabinet, bookcase and the arm chair meet it in the image plane. Again, very few lines in the scene do not belong to the LLC (the horizontal line segments of the box near the far end wall, the far arm of the armchair and two horizontal line segments attached to the book-case).

An investigation into these degeneracies and their remedy is important in the context of this thesis and the tensor brightness constraint. It is also important in the general context of 3D reconstruction from line correspondences.

1.4 Lens Distortion Calibration

The geometric constraints between two views (the epipolar constraint) and between three views (the trilinear tensors) assume a perfect pinhole camera. For medium and wide angle lenses, non-linear lens distortion and in particular radial lens distortion, can be a significant factor with errors of $10-100 pixels$ near the edges of the image [71]. These are typically the lenses used when performing image based 3D reconstruction of large objects or in a confined space, and which are used in the experiments throughout this thesis.

The investigation of the effects of lens distortion on the trilinear tensor and epipolar constraints leads to a new method for calibration of lens distortion. In this method only point correspondences between either two or three views are required. There is no need to know the 3D location of the points, the camera locations, or other internal parameters of the camera such as the focal length. In fact one can use the same point correspondence data which is to be used for subsequent vision tasks such as 3D

reconstruction. The method is therefore suitable as an online preprocessing stage.

The simulation results show that the geometric configuration of the cameras is important for the success of the calibration. The cameras must be far apart with large angles between the camera views (vergence angles). When the angles between the camera views are wide (20^o or more) then the lens distortion parameters can be recovered accurately. When the angle between views is small, as would typically be the case in nearby frames of a video sequence, the geometric constraints hold even with lens distortion. The recovered motion and structure however, will have some non-projective distortion. If we neglect the radial distortion, large planar surfaces do not appear flat in the reconstruction, and the 3D reconstructions from multiple stereo pairs do not align correctly.

1.5 An Example Application

A preview of some of the results from Chapter 5 will provide a flavor for some of the possible applications. In Chapter 5 we apply these techniques to the problem of recovering motion and extended 3D structure from a stereo image sequence. A sequence of 50 stereo pairs was used. Figures 1-4a,c show the first image pair from the sequence. Figure 1-4b shows the image from the same camera as Figure 1-4a but taken at the next time step. This sequence was taken in a laboratory but a similar set of images might be obtained from a low flying helicopter in a confined space.

From the three images (Figures 1-4a,b,c) we recover the *camera motion* and a *dense depth map* (Figure 1-4d). For a mobile robot or helicopter, the camera motion estimates can be used to augment an internal navigation system and the depth map can be used for obstacle avoidance. Edges in the depth map (Figure 1-4e) are places of *depth discontinuities*. In a traffic scene, these are places where a car or pedestrian might appear without warning. In a search-and-rescue operation these indicate places in the image where things are hidden from view. We can use this information to guide the search.

Figures 1-5a,b show 3D renderings of the reconstructed scene. The camera path shows that the camera traveled in a horizontal plane twice the height of the trees. A single image triplet can provide us with only a limited view of the scene. We can use the recovered camera motion as a way to combine multiple 3D views together to form an *extended 3D scene reconstruction* (Figure 1-6). This is an example of a *3D mosaic*. It provides a compact representation of all the information in the 50 input images and can thus be viewed as an *image compression scheme*.

1.6 Overview of the Thesis

The introduction in each chapter discusses its place in the context of the thesis and in the context of related and previous work. In the next chapter (Chapter 2) we summarize the common mathematical background needed in the rest of the thesis. We describe the projective camera model and its relationship to the perspective and

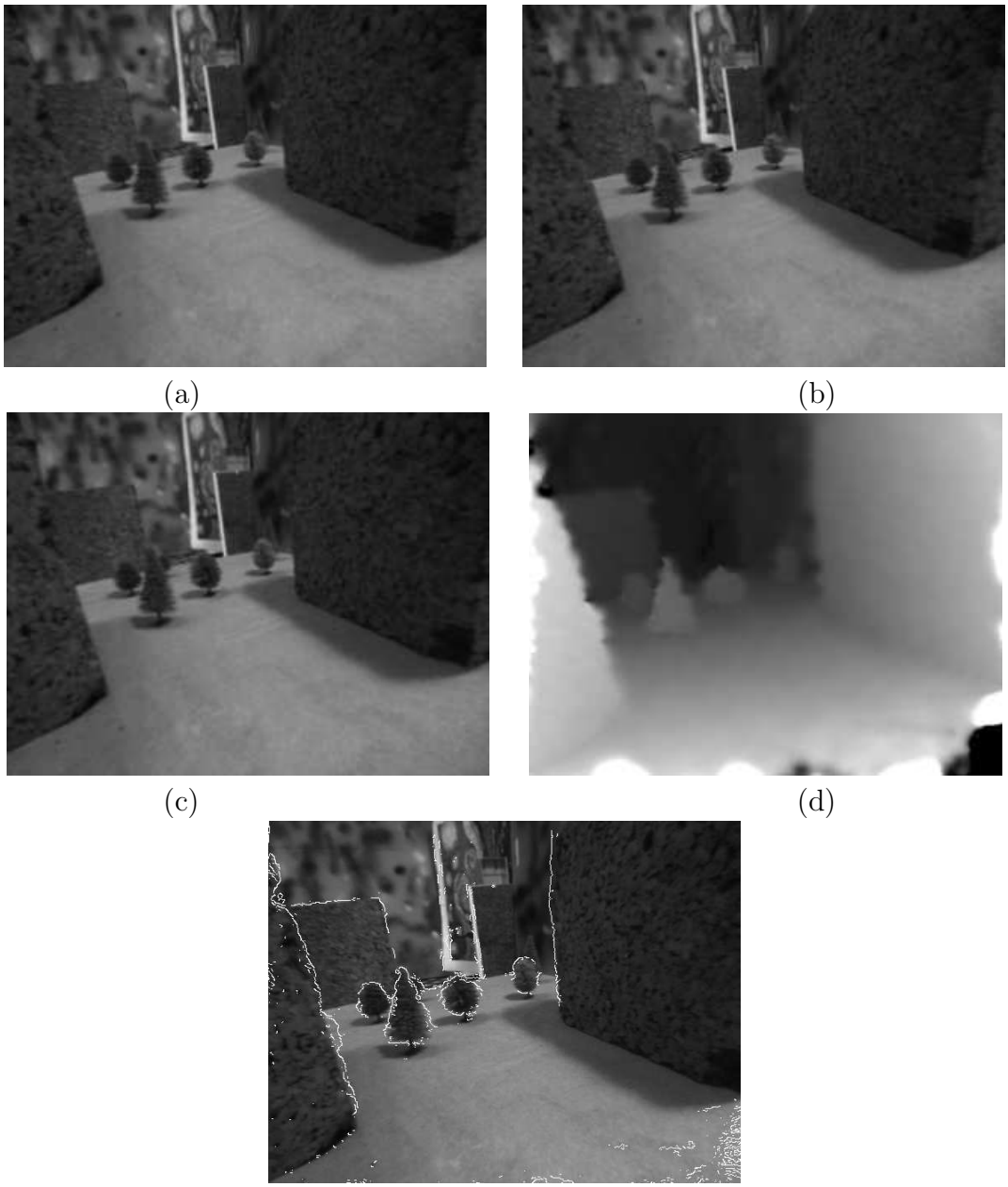


Figure 1-4: *The three input images (a,b,c). Image (a) and (c) are the first stereo pair in the sequence with the cameras displaced vertically. Image (b) is the image from camera (a) taken at the next time instance. (d) the estimated depth map. (e) Edges in the depth map identify occluding contours.*

(a) (b)

Figure 1-5: *3D renderings of the scene. Very distant surfaces have been removed. The spheres show the motion of the stereo rig in the whole sequence. The large sphere is the the location of the camera when this image was taken. Note in view (b) that the ground plane is correctly found to be flat and at right angles to the two cork walls.*

orthographic cameras. We then develop the constraint equations among point and line correspondences in two and three views. These are well known results based mainly on the work of Faugeras [16] for two views, and on Shashua's work on the trilinear tensor constraint for 3 views [57].

Chapters 3 through 5 should be read in sequence. Chapter 3 combines photogrammetric constraints in the form of the optical flow constraint equation [33] with the geometric constraint of the trilinear tensor to develop the tensor brightness constraint. We then develop simpler 'model-based brightness constraints' and describe a method for recovering structure and motion directly from image gradients. Section 3.2 derives the projective form of the constraint. Those uncomfortable with projective geometry can skim through that section and focus more on Section 3.4 and those that follow. As we have noted earlier, the optical flow constraint equation is only valid for infinitesimally small motions. In practice, this requires that the image motion be on the order of one pixel. In Section 3.7, among other details, we show how using a coarse-to-fine implementation, our method can handle displacements averaging up to 50 pixels for 640×480 resolution images. In Chapter 4 we show quantitative and qualitative results with real images.

In Chapter 5 we use the same three-view geometric constraints as a way of combining information from multiple stereo images obtained by a moving stereo rig. In addition to augmenting depth-from-stereo with additional depth information obtained from motion, this method greatly simplifies the process of bringing the depth maps from multiple stereo pairs into one coordinate frame. An example application is shown for direct estimation of motion and extended scene structure from a stereo image sequence.

In Chapter 6 we return to the trilinear tensor constraint and prove that for line correspondences the linear solution is degenerate when all the lines come from a *Linear Line Complex* (i.e. when all the lines intersect a single line in space). This

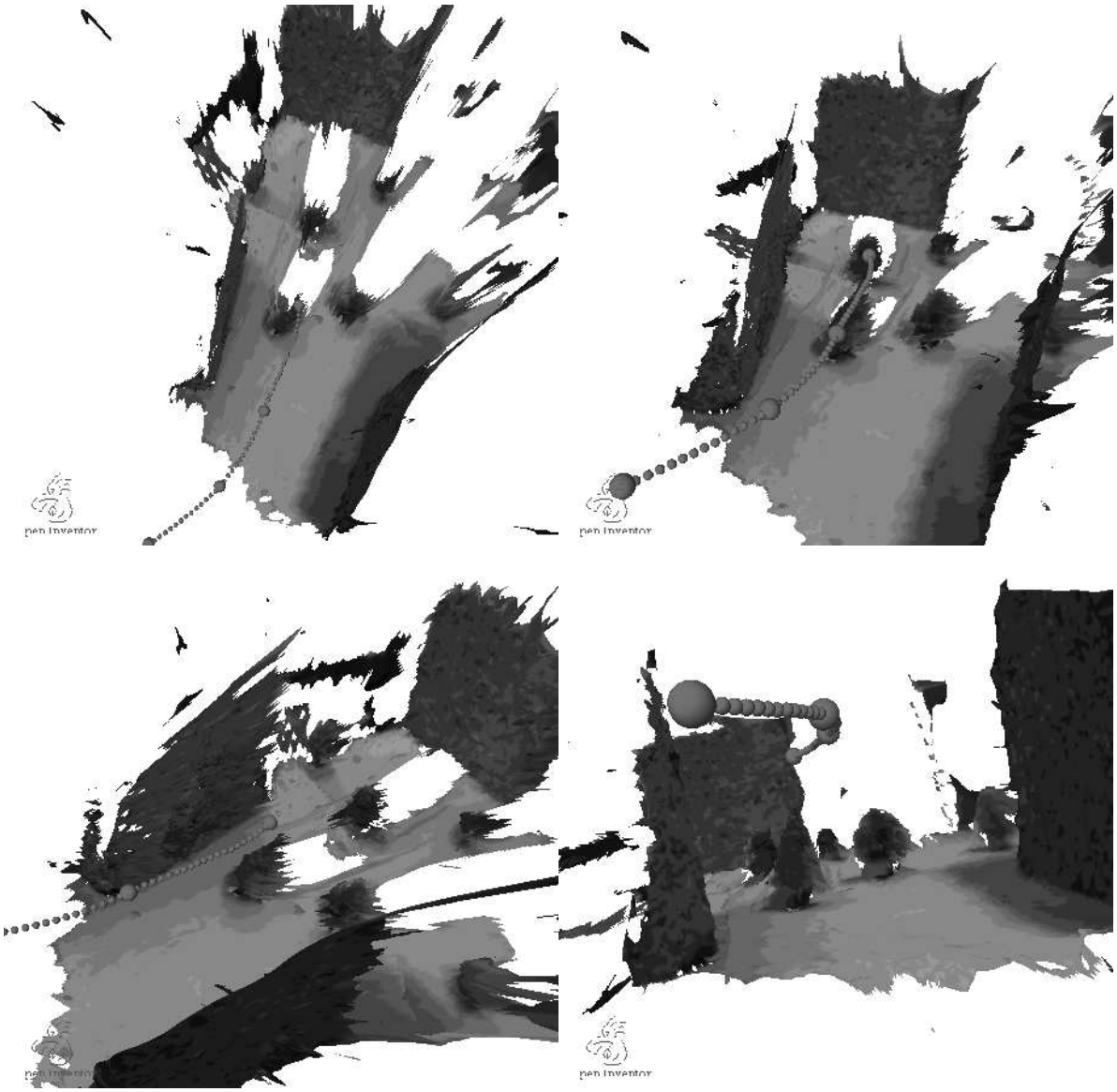


Figure 1-6: *Euclidean reconstruction of the extended scene created by aligning four separate reconstructions into one coordinate system using the recovered camera motion. The small spheres show the camera path. The large spheres show the camera positions of the four Euclidean reconstructions. There are still 'holes' in the 3D view because there are parts of the world which are not viewable from any image.*

configuration is shown to be quite common in practice, and in particular it arises in the context of the tensor brightness constraint (Chapter 3). We show a method for overcoming the degeneracy and experimental results.

The geometric constraints developed in Chapter 2 assume a perfect pinhole camera. Non-linear lens distortion and in particular radial lens distortion is a significant factor in medium and wide angel lenses and, can affect the accuracy of motion and structure estimates. Chapter 7 describes the non-linear lens distortion model and describes a new method for calibrating the lens distortion parameters using only point correspondences among multiple views. We verify the new method with simulations and experiments with real images.

Chapter 2

Mathematical Background

In this chapter we summarize the common mathematical background needed for the rest of the thesis. We first review the perspective camera model and the geometric constraint equations among point and line correspondences in two and three views. Then we review the photometric constraints, which take the form of the optical flow constraint equation.

This thesis uses results from projective geometry. A very readable introduction to the subject of projective geometry is given in the book by Young [89]. A modern book dealing more specifically with the application of projective geometry to computer vision is [16]. The appendix of [48] provides a concise summary of the basic material.

2.1 Notation

We will use uppercase bold to denote matrices (e.g. $\mathbf{A}$). Vectors representing 3D points which will be upper case and not bold. Other vectors and scalars will be in lower case.

We will use ψ to denote the first camera image or view. In the text it might also be called Image 1. The second and third views (camera images) will be called Image 2 and Image 3 respectively and denoted ψ' and ψ'' where the apostrophe $'$, and double apostrophe $''$, is used to denote variables associated with Image 2 and Image 3 respectively. Following this notation a point in Image 1, Image 2 and Image 3 will be denoted p, p' and p'' respectively. Translation from camera position of Image 1 to the camera position of Image 2 will be denoted t', translation from camera position of Image 1 to the camera position of Image 3 will be denoted t'', and so forth.

2.1.1 Points and Lines

Following [16] we use projective coordinates. A point in three-dimensional space is represented by a point in the projective space $\mathcal{P}^3$, $P \cong (X, Y, Z, W)^\top$. ($\cong$ indicates an equality up to a scale factor). A point in the image plane is represented by a point in $\mathcal{P}^2$, $p \cong (x, y, w)^\top$. The dual of a point in $\mathcal{P}^2$, the line s, is also represented by a set of three homogeneous coordinates: $s \cong (s_1, s_2, s_3)^\top$. A point p lies on a line s iff:

$$s^\top p = 0.$$

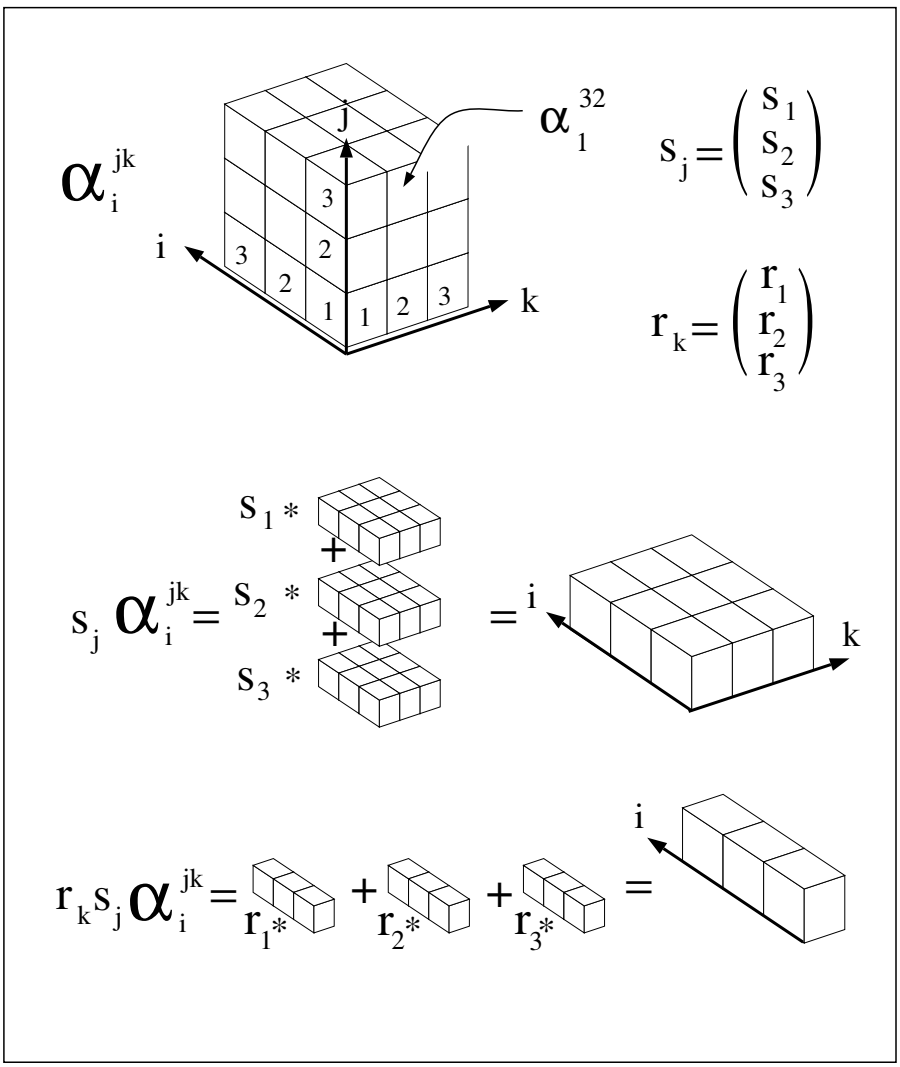


Figure 2-1: *Tensor Notation*

2.1.2 Tensor Notation

We will occasionally use tensorial notations. We use the covariant-contravariant summation convention: a point is an object whose coordinates are specified with superscripts, i.e., $p^i = (p^1, p^2, ...)$. These are called contravariant vectors. An element in the dual space (representing hyperplanes — lines in $\mathcal{P}^2$), is called a covariant vector and is represented by subscripts, i.e., $s_j = (s_1, s_2, ...)$. Indices repeated in covariant and contravariant forms are summed over, i.e., $p^i s_i = p^1 s_1 + p^2 s_2 + ... + p^n s_n$. This is known as a contraction. When using this notation we will use the indices i, j, k to denote features such as points and lines in Image 1, Image 2 and Image 3 respectively.

Figure 2-1 shows how the contraction of a 1-valence tensor (a vector) with a 3-valence tensor produces a 2-valence tensor (a matrix). A contraction of a vector with a matrix produces a vector and finally, the contraction of two 1-valence tensors is simply the familiar vector dot product.

An outer-product of two 1-valence tensors (vectors), $a_i b^j$, is a 2-valence tensor (matrix) c_i^j whose i, j entries are $a_i b^j$ — note that in matrix form $\mathbf{C} = ba^\top$.

2.2 Imaging Geometry and Motion Models

2.2.1 Perspective and Projective Cameras

We use the classic pinhole camera model, details of which can be found in [31]. A real lens has non-linear geometric distortion. We will defer the issue of lens distortion to Chapter 7 and for now assume the lens distortion, if significant, can be calibrated and compensated for.

Since the image plane is finite we can assign the value 1 to the third homogeneous coordinate thus: $p \cong (x, y, 1)^\top$. The perspective projection of the point P onto the image can be written as a linear projective equation:

$$p \cong \mathbf{M}\mathcal{P}\mathbf{K}P. \tag{2.1}$$

The 4×4 matrix $\mathbf{K}$ is the Euclidean transformation from the world coordinate system to a 3D coordinate system centered at the optical center of the camera:

$$\mathbf{K} = \begin{pmatrix} \mathbf{R} & t \\ 0_3^\top & 1 \end{pmatrix}. \tag{2.2}$$

The orthogonal rotation matrix $\mathbf{R}$ and the translation vector t are called the *extrinsic* camera parameters. The 3×4 matrix $\mathcal{P}$ is the projection matrix from the 3D camera coordinate system to the *normalized camera coordinate plane*:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.3}$$

The *normalized camera coordinate system* locates the image plane parallel to the retinal plane at a distance of one unit from the optical center. The upper triangular matrix $\mathbf{M}$ is the affine transformation from the normalized camera coordinate plane to the retinal coordinate system:

$$\mathbf{M} = \begin{pmatrix} -f_x & f_x \cot\theta & c_x \\ 0 & \frac{-f_y}{\sin\theta} & c_y \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

The values of $\mathbf{M}$ are called the *intrinsic* camera parameters. Physically, f_x and f_y are the focal length (to be precise, the principle distance) where we allow for different focal lengths in the x and y directions to take into account different spatial sampling of the image plane, and (c_x, c_y) is the principle point (i.e. the point of intersection of the optical axis of the lens and the image plane). The angle θ is the angle between the image x and y axes. With CCD cameras $\theta = 90^o$, in which case:

$$\mathbf{M} = \begin{pmatrix} -f_x & 0 & c_x \\ 0 & -f_y & c_y \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

If we know the values of the matrix $\mathbf{M}$ we refer to the camera as a *calibrated camera*. We can then move from image coordinates (p) to the normalized image coordinates (p_n) by premultiplying:

$$p_n \cong \mathbf{M}^{-1}p.$$

One often combines the matrices $\mathbf{M}\mathcal{P}\mathbf{K}$ into one 3×4 matrix:

$$p \cong \tilde{\mathbf{A}}P. \tag{2.6}$$

Note that not all 3×4 matrices are valid camera matrices but if we allow for pictures of pictures of pictures then there are no constraints on the projection matrix $\tilde{\mathbf{A}}$. This case, where there are no constraints on the matrix $\tilde{\mathbf{A}}$ is called the *projective camera*. It is clear that for any invertible matrix $\mathbf{T}$:

$$p \cong \tilde{\mathbf{A}}\mathbf{T}^{-1}\mathbf{T}P, \tag{2.7}$$

where the choice of $\mathbf{T}$ is the choice of the projective coordinate frame.

We will often split the matrix $\tilde{\mathbf{A}}$ into a 3×3 matrix (e.g. $\mathbf{A}$) and a 3×1 vector (e.g. t) which we will write as:

$$\tilde{\mathbf{A}} = [\mathbf{A}; t].$$

2.2.2 Motion Model for the Projective Case

We will present here some of the important results from [61]. Let p be the projection a 3D point P in Image 1. If we align the world coordinate system with the camera coordinate system then the (X, Y, Z) coordinates of the point $P = (X, Y, Z, 1)^\top$ are given by:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = Z\mathbf{M}^{-1}p$$

where $\mathbf{M}$ is the matrix representing the internal parameters of Camera 1. Likewise, for Image 2:

$$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = Z'\mathbf{M'}^{-1}p$$

Let $\mathbf{R}$', and t' be the relative translation and rotation between Image 1 and Image 2. Then:

$$p' \cong \mathbf{M}'\mathbf{R}'\mathbf{M}^{-1}p + \frac{1}{Z}\mathbf{M}'t'. \tag{2.8}$$

If we parameterize the plane π by its normal vector n and distance its d_π to the origin of Camera 1 then the point P lies on the plane π iff:

$$\frac{n^\top (Z\mathbf{M}^{-1}p)}{d_\pi} = 1. \tag{2.9}$$

Multiplying the right hand term of (2.8) by (2.9) leads to:

$$\begin{aligned} p' &\cong \mathbf{M}'\mathbf{R}'\mathbf{M}^{-1}p + \frac{1}{Z}\frac{1}{d_\pi}\mathbf{M}'t'\left(n^\top Z\mathbf{M}^{-1}p\right) \\ &\cong \mathbf{M}'\left(\mathbf{R}' + \frac{1}{d_\pi}t'n^\top\right)\mathbf{M}^{-1}p = \mathbf{A}p \end{aligned} \tag{2.10}$$

where $\mathbf{A}$ is the homography matrix of the plane π. We note that given the corresponding images of 4 points on the plane π, or three points and the corresponding epipoles, it is possible to find the homography matrix $\mathbf{A}$ up to a scale factor.

Let $e' = \mathbf{M}'t'$, then for a general point P:

$$\begin{aligned} p' &\cong \mathbf{M}'\mathbf{R}'\mathbf{M}^{-1}p + \frac{1}{Z}e' \\ &\cong \mathbf{A}p - \frac{1}{d_\pi}e'n^\top\mathbf{M}^{-1}p + \frac{1}{Z}e' \\ &\cong \mathbf{A}p + \left[\frac{1}{Z} - \frac{n^\top(\mathbf{M}^{-1}p)}{d_\pi}\right]e' \\ &\cong \mathbf{A}p + \left[\frac{d_\pi - n^\top(Z\mathbf{M}^{-1}p)}{Zd_\pi}\right]e' \\ &\cong \mathbf{A}p + \left[\frac{d}{Zd_\pi}\right]e' \\ &\cong \mathbf{A}p + \rho e', \end{aligned} \tag{2.11}$$

where $d = d_\pi - n^\top(Z\mathbf{M}^{-1}p)$ is the perpendicular signed distance from the point P to the plane π.

The important thing to note in the above equation is that the term in square brackets, which we have denoted ρ, is invariant to the camera parameters of the second view (i.e. it does not include any of the terms $\mathbf{R}'$, t', and $\mathbf{M}'$). The homography matrix $\mathbf{A}$, and the vector e', are each given up to a scale factor. We can set the scale of e' such that for a given point P_0, not on the plane π,:

$$p'_0 \cong \mathbf{A}p + e'.$$

With this scaling of e', for any point P:

$$\rho = \frac{Z_0}{Z}\frac{d}{d_0}$$

where d_0 is the perpendicular signed distance from the point P_0 to the plane π. When the plane π is the plane at infinity then $d, d_0 \to \infty$ and therefore $\rho \to \frac{Z_0}{Z}$ is the inverse depth, and also, since $d_\pi \to \infty$, $\mathbf{A} \to \mathbf{M}'\mathbf{R}'\mathbf{M}^{-1}$.

We now show how equation (2.10) implies that any homography matrix $\mathbf{A}_2$ can be described by any other homography matrix $\mathbf{A}_\pi$, the vector e', a scalar d and a vector $\hat{n}$:

$$\mathbf{A}_2 = \mathbf{A}_\pi + \frac{1}{d}e'\hat{n}^\top. \tag{2.12}$$

If $\mathbf{A}_\pi$ is the homography due to the plane at infinity, i.e.:

$$\mathbf{A}_\infty \cong \mathbf{M}'\mathbf{R}'\mathbf{M}^{-1},$$

then:

$$\begin{aligned} \mathbf{A}_2 &\cong \mathbf{A}_\infty + \frac{1}{d_2} e' n_2^\top \mathbf{M}^{-1} \\ &\cong \mathbf{A}_\infty + \frac{1}{d_2} e' \hat{n}_2^\top . \end{aligned} \tag{2.13}$$

Geometrically $\hat{n}_2^\top = n_2^\top \mathbf{M}^{-1}$ is the normal vector to the plane *in the camera coordinate system*. Now let $\mathbf{A}_\pi$ be the homography due to an arbitrary plane:

$$\mathbf{A}_\pi \cong \mathbf{A}_\infty + \frac{1}{d_\pi} e' \hat{n}_\pi^\top$$

and since:

$$\mathbf{A}_2 \cong \mathbf{A}_\infty + \frac{1}{d_2} e' \hat{n}_2^\top$$

we have:

$$\mathbf{A}_2 \cong \mathbf{A}_\pi + \frac{1}{d} e' \hat{n}^\top \tag{2.14}$$

for some scalar d and some vector $\hat{n}$.

For the geometric interpretation of $\hat{n}$, let L be the projection of the intersection of the planes $\mathbf{A}_\pi$ and $\mathbf{A}_2$, onto Image 1. For all points $P \in L$:

$$p' \cong \mathbf{A}_2 p \cong \mathbf{A}_\pi p.$$

Therefore, from equation (2.14) $e'\hat{n}^\top p = 0$, and $\hat{n}$ must be the projection of L in Image 1.

The Epipole

For the case where we have chosen the projective frame such that:

$$p \cong [\mathbf{I}; 0]P,$$

the vector $e' = M't'$ is the *epipole*, the projection of the camera center of the first camera in Image 2. This can be derived directly from equation (2.1) by substituting $P = (0, 0, 0, 1)^\top$:

$$e' \cong \mathbf{M}'\mathcal{P} \begin{pmatrix} \mathbf{R}' & t' \\ 0_3^\top & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{M}'t'.$$

2.2.3 Motion Model for Calibrated Cameras and Small Rotation

Calibrated Cameras

In the case of calibrated cameras we know the internal camera parameters and we can move from image coordinates (p) to the normalized image coordinates (p_n) by premultiplying:

$$p_n \cong \mathbf{M}^{-1} p.$$

Notation: when we talk about calibrated cameras we will assume that we have moved to normalized image coordinates. We will drop the n subscript and refer to normalized image points in as p, p' and p''.

Thus for calibrated cameras the projections of a point $\hat{P}$, in some arbitrary world coordinate system, into Image 1, Image 2 and Image 3 are simply:

$$\begin{array}{rcccl} p & \cong & \mathcal{P}\mathbf{K}\hat{P} & \cong & [\hat{\mathbf{R}};\hat{t}]\hat{P} \\ p' & \cong & \mathcal{P}\mathbf{K}'\hat{P} & \cong & [\hat{\mathbf{R}}';\hat{t}']\hat{P} \\ p'' & \cong & \mathcal{P}\mathbf{K}''\hat{P} & \cong & [\hat{\mathbf{R}}'';\hat{t}'']\hat{P}. \end{array}$$

By using a rigid transformation of the world coordinates of the form:

$$P = \mathbf{T}^{-1}\hat{P}$$

where:

$$\mathbf{T} = \begin{pmatrix} \hat{\mathbf{R}}^{-1} & -\hat{\mathbf{R}}^{-1}\hat{t} \\ 0 & 1 \end{pmatrix},$$

we align the world coordinate system with the coordinate of the first camera. Then:

$$\begin{array}{rcl} p & \cong & [\hat{\mathbf{R}};\hat{t}]\hat{P} \\ & \cong & [\hat{\mathbf{R}};\hat{t}]\mathbf{T}\mathbf{T}^{-1}\hat{P} \\ & \cong & [\mathbf{I};0]P \end{array}$$

and

$$\begin{array}{rcl} p' & \cong & [\mathbf{R}';t']P \\ p'' & \cong & [\mathbf{R}'';t'']P, \end{array}$$

where

$$[\mathbf{R}';t'] = [\hat{\mathbf{R}}';\hat{t}']\mathbf{T}$$

and

$$[\mathbf{R}'';t''] = [\hat{\mathbf{R}}'';\hat{t}'']\mathbf{T}.$$

In this coordinate system:

$$p \cong \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \cong \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \cong \begin{pmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{pmatrix}$$

and

$$P \cong \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \cong \begin{pmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \\ k \end{pmatrix} \cong \begin{pmatrix} x \\ y \\ 1 \\ k \end{pmatrix}$$

where $k = \frac{1}{Z}$ is the inverse depth. The motion model for calibrated cameras is therefore:

$$\begin{array}{rcl} p' & \cong & \mathbf{R}'p + kt' \\ p'' & \cong & \mathbf{R}''p + kt''. \end{array} \tag{2.15}$$

Small Rotation

If we also assume small angle rotations and can therefore make the approximations:

$$\cos(\theta) \approx 1, \quad \sin(\theta) \approx \theta,$$

the rotation matrices $\mathbf{R}'$ and $\mathbf{R}''$ become:

$$\begin{array}{rcl} \mathbf{R}' & \approx & [\mathbf{I} + [w']_x] \\ \mathbf{R}'' & \approx & [\mathbf{I} + [w'']_x], \end{array}$$

where w', w'' are the angular velocity vectors and $[\cdot]_x$ is the skew-symmetric matrix of vector products:

$$[w']_x = \begin{bmatrix} 0 & -w'_3 & w'_2 \\ w'_3 & 0 & -w'_1 \\ -w'_2 & w'_1 & 0 \end{bmatrix} \qquad [w'']_x = \begin{bmatrix} 0 & -w''_3 & w''_2 \\ w''_3 & 0 & -w''_1 \\ -w''_2 & w''_1 & 0 \end{bmatrix}.$$

The motion model for calibrated cameras and small rotation is therefore:

$$\begin{array}{rcl} p' & \cong & [\mathbf{I} + [w']_x]p + kt' \\ p'' & \cong & [\mathbf{I} + [w'']_x]p + kt''. \end{array} \tag{2.16}$$

The image motion is given by:

$$\begin{array}{rcl} p' - p & \cong & [w']_x p + kt' \\ p'' - p & \cong & [w'']_x p + kt''. \end{array} \tag{2.17}$$

2.2.4 Small Motion Model

We will now develop the equations of image motion due to the infinitesimal motion of a calibrated camera. In this section we will not be using homogeneous coordinates. Let $P = (X, Y, Z)^\top$ be the coordinates of a point in the 3D world. The motion of the point P due to a rotational velocity, $w = (w_1, w_2, w_3)^\top$, and translational velocity, $t = (t_1, t_2, t_3)^\top$, is given by:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \times \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \tag{2.18}$$

We will use $p = (x, y)$ to denote the image of the point $\tilde{P}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{X}{Y} \\ \frac{Y}{X} \end{pmatrix} \tag{2.19}$$

The motion of the image point p is given by:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left(\frac{X}{Z}\right) \\ &= \frac{\dot{X}}{Z} - \frac{X\dot{Z}}{Z^2} \\ &= \left(\frac{t_1}{Z} + w_2 - w_3\frac{Y}{Z}\right) - \frac{X}{Z}\left(\frac{t_3}{Z} + w_1\frac{Y}{Z} - w_2\frac{X}{Z}\right) \end{aligned} \tag{2.20}$$

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left(\frac{Y}{Z}\right) \\ &= \frac{\dot{Y}}{Z} - \frac{Y\dot{Z}}{Z^2} \\ &= \left(\frac{t_2}{Z} - w_1 + w_3\frac{X}{Z}\right) - \frac{Y}{Z}\left(\frac{t_3}{Z} + w_1\frac{Y}{Z} - w_2\frac{X}{Z}\right) \end{aligned} \tag{2.21}$$

Noting that $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$, and rearranging the terms we get the infinitesimal motion equations of Longuet-Higgins & Prazdny [42]:

$$\begin{aligned} u &= \frac{1}{z}(t_1 - xt_3) - w_3 y + w_2(1 + x^2) - w_1' xy \\ v &= \frac{1}{z}(t_2 - yt_3) + w_3 x - w_1(1 + y^2) + w_2' xy \end{aligned} \tag{2.22}$$

where we use $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$. For discrete time steps: $u' = x' - x$ and $v' = y' - y$.

2.2.5 Weak Perspective, Orthographic and Affine

Weak Perspective

We will consider the case of a simple set of internal camera parameters:

$$\mathbf{M} = \begin{pmatrix} -f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.23}$$

In this case the camera matrix $\tilde{\mathbf{A}}$ is given by:

$$\tilde{\mathbf{A}} = \begin{pmatrix} -fR_1 & -ft_1 \\ -fR_2 & -ft_2 \\ R_3 & t_3 \end{pmatrix} \tag{2.24}$$

where R_i denotes the i'th row of the rotation matrix $\mathbf{R}$. A set of points $P_i = (X_i, Y_i, Z_i, 1)^\top \in \mathcal{P}^3$ project to the points $p_i = (x_i, y_i, 1)^\top \in \mathcal{P}^2$. Let $\tilde{P}_i = (X_i, Y_i, Z_i)^\top \in \mathcal{R}^3$ denote the coordinates of the points in Euclidean coordinates.

$$p_i \cong \begin{pmatrix} -fR_1\tilde{P}_i - ft_1 \\ -fR_2\tilde{P}_i - ft_2 \\ R_3\tilde{P}_i + t_3 \end{pmatrix} \tag{2.25}$$

Now let us assume that the distance to the centroid of set of points P_0 is large compared to the variation in depth among the points. If $P_i = P_0 + \delta_i$ this assumption translates to:

$$\frac{R_3\delta_i}{R_3P_0} << 1. \tag{2.26}$$

Using this assumption we can write equation (2.25) as:

$$p_i \cong \begin{pmatrix} -fR_1\tilde{P}_i - ft_1 \\ -fR_2\tilde{P}_i - ft_2 \\ R_3\tilde{P}_0 + t_3 \end{pmatrix}. \tag{2.27}$$

The third element of the projective image point p_i is the same for all the points. From equation (2.27) under this assumption $\tilde{\mathbf{A}}$ has the form:

$$\tilde{\mathbf{A}}_{\mathbf{wp}} = \begin{pmatrix} -fR_1 & -ft_1 \\ -fR_2 & -ft_2 \\ 0 & R_3\tilde{P}_0 + t_3 \end{pmatrix}. \tag{2.28}$$

This projection is called *weak perspective*. It preserves parallelism since a point $P_\infty = (X, Y, Z, 0)$ on the plane at infinity is projected to point $p_\infty = (x, y, 0)$ on the line at infinity on the image plane. We will now explore how the image of a set of points changes as the camera moves.

Translation: We can see from equation (2.28) that motion in the depth (translation along the optical axis) will amount to a uniform scaling of the whole image. The value of the scaling depends on the depth to the centroid of the points P_0. Translation along the X or Y axis will cause a uniform shift of the image. This is in contrast to perspective projection where the image motion of each point depended on the depth of that point. In weak perspective translation does not cause parallax and does not give information about relative depth.

Rotation: Rotation around the optical axis is simply a rotation of the image just like in the perspective case. For rotation around the Y axis the rotation matrix R is given by:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

which results in the projection matrix:

$$\tilde{\mathbf{A}}_{\mathbf{wp}} = \begin{pmatrix} f\cos\theta & 0 & f\sin\theta & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta X_0 + \cos\theta Z_0 \end{pmatrix}. \tag{2.29}$$

The motion of point in the image due to rotation depends on the depth. This is very different from perspective projection where camera rotation gives no depth information.

Does this Contradict the Perspective Model? We have found that in weak perspective, translation appears to give no depth information while rotation does.

This appears to contradict the perspective model where translation provides depth information through parallax while rotation gives no depth information. If we look carefully at the weak perspective equation (2.29) we notice that the image motion due to rotation around the Y axis, neglecting the effects of the overall scale change, is given by:

$$u'_i = x'_i - x_i \approx \sin\theta \frac{Z_i}{Z_0} = \sin\theta(1 + \frac{\delta_i}{Z_0}).$$

But $Z_0 >> \delta_i$ and therefore the variation in image flow over the set of points will be small (and that is where the depth or shape information lies). To get a large variation we need to use large angles of rotation and then the term $\sin\theta(1 + \frac{\delta_i}{Z_0})$ will cause the set of points to disappear from inside the image. We must then translate the camera along the X axis to reposition the points inside the image. Therefore to get usable measurements we must both rotate and translate the camera. From the perspective point of view we translate the camera along the X axis to get depth from parallax but since the depth variation is very small (the weak perspective assumption) we must translate a lot to get accurate shape information.In order to keep the points in view we need to rotate the camera. So according to both models, in order to get shape information from scene with little depth variation, we must have a large translation and rotation and there is no contradiction.

The Affine Camera

The *affine camera* is a generalization of weak perspective where there is no restriction on the elements of the projection matrix other than that the first three element of the third row are zero:

$$\tilde{\mathbf{A}}_{\mathbf{affine}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \end{pmatrix}.$$

The affine projection also preserves the plane at infinity and therefore preserves parallelism. We can use this model if we do not know the internal camera parameters.

Orthographic Projection

Orthographic projection is a specialization of the weak perspective to the case where the centroid of the points is infinitely far away but the focal length (f) is infinitely large. Thus the scale factor is constant and does not change with motion along the optical axis:

$$\tilde{\mathbf{A}}_{\mathbf{ortho}} = \begin{pmatrix} R_1 & t_1 \\ R_2 & t_2 \\ 0 & 1 \end{pmatrix}. \tag{2.30}$$

2.3 The Geometry of Two Views: the Epipolar Constraint

We will consider two perspective views ψ and ψ' of a 3D scene in Camera 1 and Camera 2 respectively. The centers of projection of the two cameras will be labeled O_1 and O_2 respectively. Let P be a 3D point in $\mathcal{R}^3$. Let P_1 and P_2 be the coordinates of the point P in the coordinate systems of Camera 1 and Camera 2 . The two camera coordinates frames are related through the Euclidean transformation:

$$P_1 = \mathbf{R}P_2 + T$$

where $\vec{T} = \vec{O_1O_2}$ is the coordinate vector of O_2 in the coordinate frame of Camera 1.

The vectors $\vec{O_1P}$, $\vec{O_2P}$ and $\vec{T}$ are coplanar. Therefore in coordinate frame of Camera 1:

$$\vec{O_1P} \cdot \left(\vec{T} \times \vec{O_2P}\right) = 0. \tag{2.31}$$

The cross product by T can be written using the skew symmetric matrix:

$$[t]_x = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}.$$

Substituting $\vec{O_1P} = P_1$, $\vec{O_2P} = \mathbf{R}P_2 + T$ in equation (2.31) we get:

$$P_1^\top \mathbf{E} P_2 = 0 \tag{2.32}$$

where $\mathbf{E} = [t]_x\mathbf{R}$ is called the *essential matrix* [41]. The normalized image coordinates p_1 and p_2 are proportional to the 3D Cartesian point coordinates P_1 and P_2 thus:

$$p_1^\top \mathbf{E} p_2 = 0. \tag{2.33}$$

The above equation was developed for the Euclidean case. The projective case can be obtained by applying a 3×3 planar projective transformation to the normalized image coordinates of each camera:

$$p_1 = \mathbf{H}_1 p_1'$$

and

$$p_2 = \mathbf{H}_2 p_2'.$$

Equation (2.33) can then be written in the form:

$$p_1'^\top \tilde{\mathbf{H}}_1 \mathbf{E} \mathbf{H}_2 p_2' = p_1'^\top \mathbf{F} p_2' = 0 \tag{2.34}$$

where $\tilde{\mathbf{H}}_1 = \mathbf{H}_1^{(-1)}$, and $\mathbf{F} = \tilde{\mathbf{H}}_1 \mathbf{E} \mathbf{H}_2$ is called the *fundamental matrix* [16].

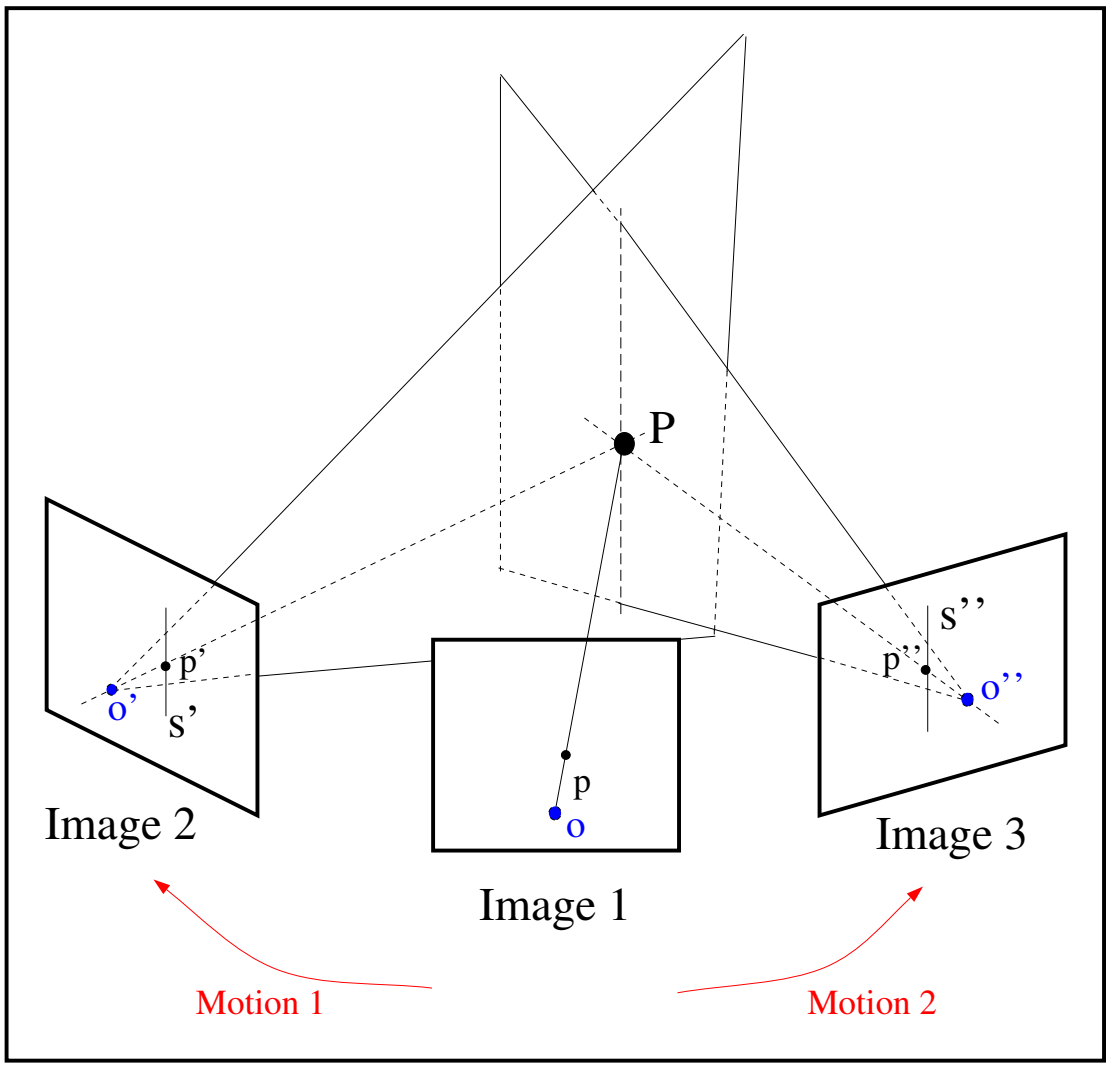


Figure 2-2: *The Geometry of Three Views: Image 1, Image 2 and Image 3 are three views obtained from cameras centered at o, o' and o'' respectively. Point P in the scene projects to image points p, p' and p''. s' and s'' are any image lines passing through the points p' and p'' respectively.*

2.4 The Geometry of Three Views - the Trilinear Tensor

We will now consider three perspective views ψ, ψ' and ψ'' of a 3D scene. Figure 2-2 shows a 3D point $P \in \mathcal{P}^3$ and its image in the three views $p \in \psi$, $p' \in \psi'$ and $p'' \in \psi''$.

Without loss of generality we can align the 3D world coordinate system with the coordinate system of the first camera:

$$p \;\cong\; \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \;\cong\; [\mathbf{I}; 0]\, P \tag{2.35}$$

Thus $P \cong (x, y, 1, \rho)^\top$. After we have set the first camera coordinate system, the other two camera coordinate systems are relative to the first. From equation (2.11):

$$\begin{array}{rcccccccc} p' & \cong & \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} & \cong & \tilde{\mathbf{A}}P & \cong & [\mathbf{A}; t']\, P & \cong & \mathbf{A}p + \rho t' \\ p'' & \cong & \begin{pmatrix} x'' \\ y'' \\ 1 \end{pmatrix} & \cong & \tilde{\mathbf{B}}P & \cong & [\mathbf{B}; t'']\, P & \cong & \mathbf{B}p + \rho t'' \end{array} \tag{2.36}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are homography matrices from Image 1 to Image 2 and to Image 3 respectively *due to the same plane* π. The vectors t' and t'' are the epipoles, the projection center of camera 1 projected onto the image planes of the 2nd and 3rd cameras respectively. In a calibrated setting, (i.e. if the intrinsic parameters are known), the matrices $\mathbf{A}$ and $\mathbf{B}$ are rotations and the vectors t' and t'' are the translations. Finally, in the calibrated setting ρ is replaced by $k = \frac{1}{z}$.

It has been shown [57] that the image coordinates of a point in three views are related through 4 trilinear equations. The basic idea can be neatly summarized in the form of the *trilinear tensor* in the following proposition.

Proposition 1 (The trilinear tensor) *Let P be a point in the projective space $\mathcal{P}^3$ and let its image in three views be p, p', p''. Let s' and s'' be any lines in the image plane passing through the image points p' and p'' respectively. p, s' and s'' are related through the following equation:*

$$p^i s''_k s'_j \mathcal{T}_i^{jk} = 0$$

Proof:

Let $P \cong (x, y, 1, \rho)^\top$ be a point in the projective space $\mathcal{P}^3$ and its image in 3 views, p, p', p''. Let s' and s'' be lines through points p' and p'' respectively:

$$s'^\top p' = 0 \qquad\qquad s''^\top p'' = 0 \tag{2.37}$$

As we recall from equation (2.36):

$$\begin{array}{rcl} p' & \cong & \mathbf{A}p + \rho t' \\ p'' & \cong & \mathbf{B}p + \rho t'' \end{array}$$

Premultiplying the left and right hand sides of the first equation by $s'^{\top}$ and the second equation by $s''^{\top}$ we get:

$$\begin{array}{rcl} s'^{\top}\mathbf{A}p + \rho s'^{\top}t' & = & 0 \\ s''^{\top}\mathbf{B}p + \rho s''^{\top}t'' & = & 0 \end{array} \tag{2.38}$$

Eliminating ρ from the above equations results in the equation:

$$s'^{\top}t's''^{\top}\mathbf{B}p - s''^{\top}t''s'^{\top}\mathbf{A}p = 0 \tag{2.39}$$

This can be written compactly using tensor notation:

$$p^i s''_k s'_j \mathcal{T}_i^{jk} = 0 \tag{2.40}$$

where $\mathcal{T}$ is the tensor representing a bilinear function of the camera matrices:

$$\mathcal{T}_i^{jk} = t'^j b_i^k - t''^k a_i^j \tag{2.41}$$

□

Equation (2.40) relates a point p in Image 1 and lines s' and s'' passing through the corresponding points p' and p'' in Image 2 and Image 3 respectively. It is important to note that the lines s' and s'' do not have to correspond to any physical line in space or the image. Taking any two distinct lines through p' and two distinct lines through p'' gives rise to four distinct trilinear equations. Any others lines through p' or p'' are simply linear combinations of the previous ones and therefore give rise to no new constraints. We can write equations relating point correspondences between three views if we choose the *standard* form where s' (and s'') represent either the vertical or horizontal scan lines, i.e.,

$$s' = \begin{pmatrix} -1 \\ 0 \\ x' \end{pmatrix} \quad or \quad s' = \begin{pmatrix} 0 \\ -1 \\ y' \end{pmatrix}$$

and

$$s'' = \begin{pmatrix} -1 \\ 0 \\ x'' \end{pmatrix} \quad or \quad s'' = \begin{pmatrix} 0 \\ -1 \\ y''. \end{pmatrix}$$

Then the four trilinear forms, referred to as *trilinearities* [57], have the following explicit form:

$$\begin{array}{l} x''\mathcal{T}_i^{13}p^i - x''x'\mathcal{T}_i^{33}p^i + x'\mathcal{T}_i^{31}p^i - \mathcal{T}_i^{11}p^i = 0, \\ y''\mathcal{T}_i^{13}p^i - y''x'\mathcal{T}_i^{33}p^i + x'\mathcal{T}_i^{32}p^i - \mathcal{T}_i^{12}p^i = 0, \\ x''\mathcal{T}_i^{23}p^i - x''y'\mathcal{T}_i^{33}p^i + y'\mathcal{T}_i^{31}p^i - \mathcal{T}_i^{21}p^i = 0, \\ y''\mathcal{T}_i^{23}p^i - y''y'\mathcal{T}_i^{33}p^i + y'\mathcal{T}_i^{32}p^i - \mathcal{T}_i^{22}p^i = 0. \end{array} \tag{2.42}$$

These constraints were first derived in [57]; the tensorial derivation leading to equations (2.41) and (2.40) was first derived in [63]. The trilinear tensor has been well

known in disguise in the context of Euclidean line correspondences and was not identified at the time as a tensor but as a collection of three matrices (a particular contraction of the tensor, correlation contractions, as explained next) [68, 70, 84]. The link between the two and the generalization to projective space was identified later by Hartley [24, 25]. Additional work in this area can be found in [62, 17, 80, 30, 59, 2], and applications in [3, 74].

2.4.1 Tensor Slices and Contractions

From Figure 2-1 it is clear that the tensor can be sliced in three principle ways. This can be generalized into performing a contraction operation on the tensor with a vector. Contracting the tensor along each of the three coordinates i, j, k produces matrices each with its own distinct geometric properties.

We will mostly focus on the contraction:

$$\delta^i \mathcal{T}_i^{jk} \tag{2.43}$$

where δ^i is an *arbitrary* contravariant vector and the results of the contraction is a 3×3 matrix, which we denote by $\mathbf{T}$.

The general meaning of the matrix $\mathbf{T}$ can be found in [58] where we have the proposition:

Proposition 2 (Correlation Contractions) *The contraction $\delta^i \mathcal{T}_i^{jk}$ for some arbitrary δ^i is a rank 2 correlation matrix from Image 2 onto Image 3, that maps the dual image plane (the space of lines in Image 2) onto a set of collinear points in Image 3 that form the epipolar line corresponding to the point δ^i in Image 1. The null space of the correlation matrix is the epipolar line of δ^i in Image 2. Similarly, the transpose of $\mathbf{T}$ is a correlation from Image 3 onto Image 2 with the null space being the epipolar line in Image 3 corresponding to the point δ^i in Image 1.*

For a full proof see [58]. We will provide here a simple proof of the rank 2 constraint:

Proposition 3 *Rank($\delta^i \mathcal{T}_i^{jk}$)=2 for all choices of δ^i.*

Proof: We recall that:

$$\mathcal{T}_i^{jk} = t'^j b_i^k - t''^k a_i^j$$

Therefore:

$$\delta^i \mathcal{T}_i^{jk} = \sum_{i=1}^{3} \delta^i (t' b_i^\top - t'' a_i^\top) \tag{2.44}$$

where a_i and b_i are the i'th columns of the matrices $\mathbf{A}$ and $\mathbf{B}$ respectively. This can be rewritten as:

$$\delta^i \mathcal{T}_i^{jk} = t' \sum_{i=1}^{3} (\delta^i b_i^\top) - t'' \sum_{i=1}^{3} (\delta^i a_i^\top) \tag{2.45}$$

The outer product of two vectors is a matrix of $rank = 1$ and the matrix $\delta^i \mathcal{T}_i^{jk}$ is seen to be a linear combination of two matrices of $rank = 1$ and is therefore in general of $rank = 2$. □

We can define the *Standard Correlation Slicing* as the correlation contractions associated with selecting δ be $(1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$, thus the three standard correlation slices are $\mathcal{T}_1^{jk}, \mathcal{T}_2^{jk}$ and $\mathcal{T}_3^{jk}$, and we denote them by $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, respectively. The three standard correlations date back to the work on structure from motion of lines across three views [68, 84] where these matrices were first introduced.

We note also the contraction:

$$\mathbf{E} = \delta_k \mathcal{T}_i^{jk} = \delta_1 \mathcal{T}_i^{j1} + \delta_2 \mathcal{T}_i^{j2} + \delta_3 \mathcal{T}_i^{j3} \tag{2.46}$$

which results in the 3×3 matrix $\mathbf{E}$, and δ_k is an *arbitrary* covariant vector. The matrix $\mathbf{E}$ has a general meaning introduced in [62]:

Proposition 4 (Homography Contractions) *The contraction $\delta_k \mathcal{T}_i^{jk}$ for some arbitrary δ_k is a homography matrix from Image 1 onto Image 2 determined by the plane containing the third camera center C'' and the line δ_k in the third image plane. Generally, the rank of $\mathbf{E}$ is 3. Likewise, the contraction $\delta_j \mathcal{T}_i^{jk}$ is a homography matrix from Image 1 onto Image 3, denoted $\mathbf{W}$.*

For proof and applications see [62].

2.4.2 Tensor Admissibility Constraints

The 27 coefficients $\mathcal{T}_i^{jk}$ are not independent. It has been shown in [15] and [58] that the tensor is determined by only 18 parameters. We will first present the explanation from [58]. We will then show how from the contraction properties discussed above, one can come up with the constraints among the 27 coefficients, referred to as *admissibility constraints*. These can are grouped into three classes. (For further details see [58]).

18 Parameters

The tensor

$$\mathcal{T}_i^{jk} = t'^j b_i^k - t''^k a_i^j$$

is determined by 24 parameters given by the two camera matrices, each has 12 parameters. Two additional parameters drop out because we can scale t' and accordingly b_i^k without changing the tensor, and likewise scale t'' and accordingly a_i^j. An additional parameter drops out because of the global scale factor (tensor is determined up to overall scale). Thus, we readily see there can be at most 21 parameters defining the tensor. We can drop out three more parameters by noticing that the matrices a_i^j and b_i^k belong to a family of homography matrices that leaves the tensor unchanged (uniqueness proof in [57]), as detailed below:

$$\mathcal{T}_i^{jk} \quad = \quad t'^j b_i^k - t''^k a_i^j \tag{2.47}$$

$$
\begin{aligned}
&= t'^j(b_i^k + \alpha_i t''^k) - t''^k(a_i^j + \alpha_i t'^j) && (2.48)\\
&= \mathcal{T}_i^{jk} + \alpha_i t'^j t''^k - \alpha_i t'^j t''^k && (2.49)\\
&= \mathcal{T}_i^{jk} && (2.50)
\end{aligned}
$$

hence, we have three free parameters α_i (in geometric terms there is a free choice of reference plane in space). We can select α_i such that the matrix b_i^k will have a vanishing column (this corresponds to selecting a reference plane coplanar with the center of projection of the third view). Therefore, the new matrices a_i^j and b_i^k have only 15 non-vanishing entries, and we have reduced the number of parameters from 21 to 18.

Admissibility Constraints

We may deduce from the Correlation Contractions discussed above the following three groups of constraints that the 27 coefficients must satisfy:

1. Rank($\delta^i \mathcal{T}_i^{jk}$)=2 for all choices of δ. The three standard correlation slices $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ are of rank 2 each and this property is *closed* under all linear combinations.

2. Rank(null($\mathbf{T}_1$),null($\mathbf{T}_2$),null($\mathbf{T}_3$))=2. This follows from the fact the the null space of $\delta^i \mathcal{T}_i^{jk}$ is the epipolar line in the second image corresponding to the point δ in the first image — since all epipolar lines are concurrent, their rank is 2.

3. Rank(null($\mathbf{T}_1^\top$),null($\mathbf{T}_2^\top$),null($\mathbf{T}_3^\top$))=2. These are epipolar lines in third image, thus their rank is 2 as well.

One can easily show that no subset of these constraints is sufficient to describe an admissible tensor of the form of equation (2.41). For example, the tensor:

$$\mathcal{T}_i^{jk} = t'^j b_i^k - t''^j a_i^k$$

where we have switched the position of the j and k indices in the $t''^j a_i^k$ term, will satisfy the first constraint but not the second.

In practice, in the presence of errors in image measurements one often starts with the linear solution (that might not satisfy the admissibility constraints) and improves it further by employing a numerical Gauss-Newton style iterative procedure until a solution that satisfies the admissibility constraints is obtained (for example, [15]).

2.5 Photometric Constraints - the Optical Flow Constraint Equation

We now move on to the photometric constraints between two images. We will derive the *optical flow constraint equation* following [31]. The basic assumption of the optical flow equation is that the irradiance (brightness) of an image point (x, y) at time t (i.e. $I(x, y, t)$), is equal to the irradiance of the corresponding image point (x', y') in

the second image taken at time t' (i.e. $I(x', y', t')$). (In this section t and t' are used to denote time not translations.) This is known as the *constant brightness constraint*. It can be written as:

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \tag{2.51}$$

where $\delta t = t' - t$, $\delta x = x' - x$ and $\delta y = y' - y$. Assuming the image brightness is smooth we can expand the left-hand-side of the equation as a Taylor series:

$$I(x, y, t) + \delta x I_x + \delta y I_y + \delta t I'_t + e = I(x, y, t) \tag{2.52}$$

where I_x, I_y, I'_t are the spatial and temporal derivatives of image brightness and e contains the higher order terms. Cancelling the terms $I(x, y, t)$ from both sides, dividing by δt and taking the limit as $\delta t \to 0$ results in the *optical flow constraint equation* of Horn & Schunk [33]:

$$u' I_x + v' I_y + I'_t = 0 \tag{2.53}$$

where:

$$u'(x, y) = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$$

and

$$v'(x, y) = \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$$

are the *optical flow* values at (x, y) (i.e. the motion of the image point in the x and y directions between frames: $u' = x' - x$ and $v' = y' - y$). In practice, I_x, I_y are the components of the image gradient at location (x, y), and I'_t is the discrete temporal derivative at (x, y), i.e., $I_2(x, y) - I_1(x, y)$, where I_2 and I_1 are the image intensity values of the second and first images, respectively.

Equation (2.53) provides a single constraint on the values (u, v) given I_x, I_y and I'_t. Values of (u, v) that satisfy this equation are constrained to lie on a straight line. The component of the flow in the direction of the brightness gradient is given by:

$$\frac{I'_t}{\sqrt{I_x^2 + I_y^2}}.$$

This is also called the *normal flow*. We cannot find the component of the flow at right angles to this direction (i.e. tangent to the iso-brightness contour). This leads to the *aperture problem* described next.

2.5.1 The Aperture Problem

Imagine a black bar moving in front of a white background and suppose we can only observe the scene through a finite window or aperture (Figure 2-3). From our limited view we can tell how much the bar has moved in the direction perpendicular to the edge (i.e. in the direction of the gradient) but we cannot tell how much motion there was, if any, along the edge of the bar. Thus our interpretation of the scene is

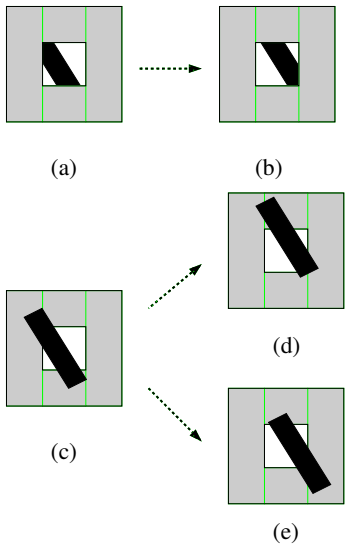


Figure 2-3: *The aperture problem: Due to the finite aperture the motion of the black bar between images (a) and (b) has an ambiguous interpretation. Two possible interpretations are the motions (c → d) and (c → e). If the window (aperture) was larger and the ends of the finite bar could be seen, this ambiguity disappears.*

ambiguous. If the window was larger and we could see the ends of the finite bar this ambiguity disappears.

Various attempts have been made to overcome the aperture problem by adding constraints on the brightness derivatives:

$$\frac{d\left(\nabla I\right)}{dt} = 0 \tag{2.54}$$

or

$$\nabla\left(\frac{dI}{dt}\right) = 0. \tag{2.55}$$

A good survey of these approaches appears in [49]. These approaches lead to additional constraint equations on the optical flow which involve second partial derivatives of the image brightness. In order to compute the optical flow it is required that the Hessian of the image brightness at time t is non singular, a condition that is not met, for example, along straight edges. Another problem with these approach is that the constraint (2.54) is only correct for surfaces that are on planes parallel to the image plane. The direction of the iso-brightness contour in the image of lines that vary in depth, does in fact change subtly. This subtlety can be important, as we will see in the 'future work' section of Chapter 3 (Section 3.8.1).

2.5.2 The Optical Flow Constraint as a Point-Line Correspondence

We now present a new interpretation of the optical flow constraint equation as a point-line correspondence between the two images (to a first order approximation).

In projective coordinates the optical flow constraint equation can be written in the form:

$$\begin{pmatrix} I_x \\ I_y \\ I'_t \end{pmatrix}^\top \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = 0 \tag{2.56}$$

The line $s = (I_x, I_y, -xI_x - yI_y)^\top$ in the projective plane passes through the point $p = (x, y, 1)^\top$ since:

$$\begin{pmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0. \tag{2.57}$$

Combining equations (2.56) and (2.57):

$$\begin{pmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{pmatrix}^\top \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + \begin{pmatrix} I_x \\ I_y \\ I'_t \end{pmatrix}^\top \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = 0. \tag{2.58}$$

Noting that $x' = x + u$ and $y' = y + v$, and rearranging the terms, equation (2.58) becomes:

$$\begin{pmatrix} I_x \\ I_y \\ I'_t - xI_x - yI_y \end{pmatrix}^\top \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0, \tag{2.59}$$

and therefore, the line:

$$s' = \begin{pmatrix} I_x \\ I_y \\ I'_t - xI_x - yI_y \end{pmatrix}^\top \tag{2.60}$$

passes through the point $p' = (x', y', 1)^\top$.

In summary: Given the coordinates of a point $p = (x, y, 1)^\top$ and the spatial and temporal derivatives (I_x, I_y, I'_t), the optical flow constraint equation gives us the equation of a line s' that passes through the corresponding point $p' = (x', y', 1)^\top$ (equation 2.60). Thus, we have a correspondence between a point in one image a line passing through the corresponding point in the second image. We do not need to know the coordinates of the point in the second image.

2.6 Summary

Geometric Constraints

The main result from this section which will be used throughout the rest of the thesis is the trilinear tensor equation:

$$p^i s''_k s'_j \mathcal{T}_i^{jk} = 0$$

where p is a point in Image 1 and s', s'' are lines going through the corresponding points p', p'' in Image 1 and Image 2 respectively. The $3\times3\times3$ tensor $\mathcal{T}_i^{jk}$ encapsulates the camera motions and internal camera parameters:

$$\mathcal{T}_i^{jk} = t'^j b_i^k - t''^k a_i^j.$$

The 27 coefficients of the tensor are not algebraically independent and there are in fact only 18 algebraically independent parameters. For a given set of 27 coefficients to be a valid tensor they must obey a set admissibility constraints.

In general, 7 corresponding points or 13 corresponding lines, in 3 views, give enough equations in form of equation (2.40) or equation (2.42) to solve for the tensor coefficients using linear methods. One can then refine the tensor coefficients to enforce the algebraic constraints [15]. As we show in Chapter 6 there are certain line configurations where the linear solution is degenerate but the admissibility constraints enable us to arrive at a unique solution.

Photometric Constraints

Given the coordinates of a point p and the spatial temporal derivatives (I_x, I_y, I'_t), the optical flow constraint equation provides a first order approximation of the equation of a line s' passing through the corresponding point p'. The line s' is given by:

$$s' = \begin{pmatrix} I_x \\ I_y \\ I'_t - xI_x - yI_y \end{pmatrix}^\top \tag{2.61}$$

Chapter 3

Model-based Brightness Constraints: Direct Estimation of Motion and Structure

3.1 Introduction

In this chapter we combine the geometric constraints of the motion models from Chapter 2 with the photometric constraints to form *model-based brightness constraints*. The result is a new method for computing the camera motion (ego-motion) between three views in an image sequence. The motion estimates are then used to compute dense structure. This method can be viewed as an extension of the 'direct methods' of Horn & Weldon [34] from two views (one motion) to three views (two motions). These methods are dubbed 'direct methods' because they do not require prior computation of optical flow.

Why three views (two motions) and not two views (one motion)? As we have seen in Section 2.5, the photometric constraints, in the form of the optical flow constraint equation, provide a matching constraint between each point in one image and a line passing through the corresponding point in the second image. In other words, for every point in one image it gives us the equation of a line along which the corresponding point must lie in the second image. There are no geometric constraints on point-line correspondences between two views. We must therefore use three view geometry, where a point in one view, and lines through the corresponding points in two other views, provide a constraint, which can be written in the form of the *trilinear tensor* (Section 2.4).

We can also use a counting argument. A single image pair provides one equation for each point in the image for a total of N equations, where N is the number of points in the image. There are $N + 5$ unknowns: the N unknown depths and the 3 translation and 3 rotation parameters with one unknown dropping out because translation and depth can only be found up to a common scale factor. The problem is therefore ill-posed. Adding another motion provides an additional N equations while adding only 6 unknowns, the second translation and rotation parameters.

By combining the optical flow constraint (Section 2.5) with the trilinear tensor model (Section 2.4) we obtain a model-based brightness constraint. The constraint describes the relationship between the spatio-temporal derivatives at each pixel in

the image with the unknown ego-motion parameters modeled by the 27 coefficients of the trilinear tensor. This constraint, the *tensor brightness constraint*, provides a linear constraint per pixel in the image which results in a highly over-constrained set of equations. We then proceed through a hierarchy of reduced motion models first by assuming calibrated cameras and small rotations, and then by assuming the Longuet-Higgins & Prazdny small motion model [42], resulting in reduced model-based brightness constraints for those motion models. Since these methods are based on the image gradients and the 'constant brightness constraint', they are only exact for infinitesimal image motion. By using a coarse-to-fine implementation our method can handle displacements averaging up to 50 pixels for 640×480 resolution images.

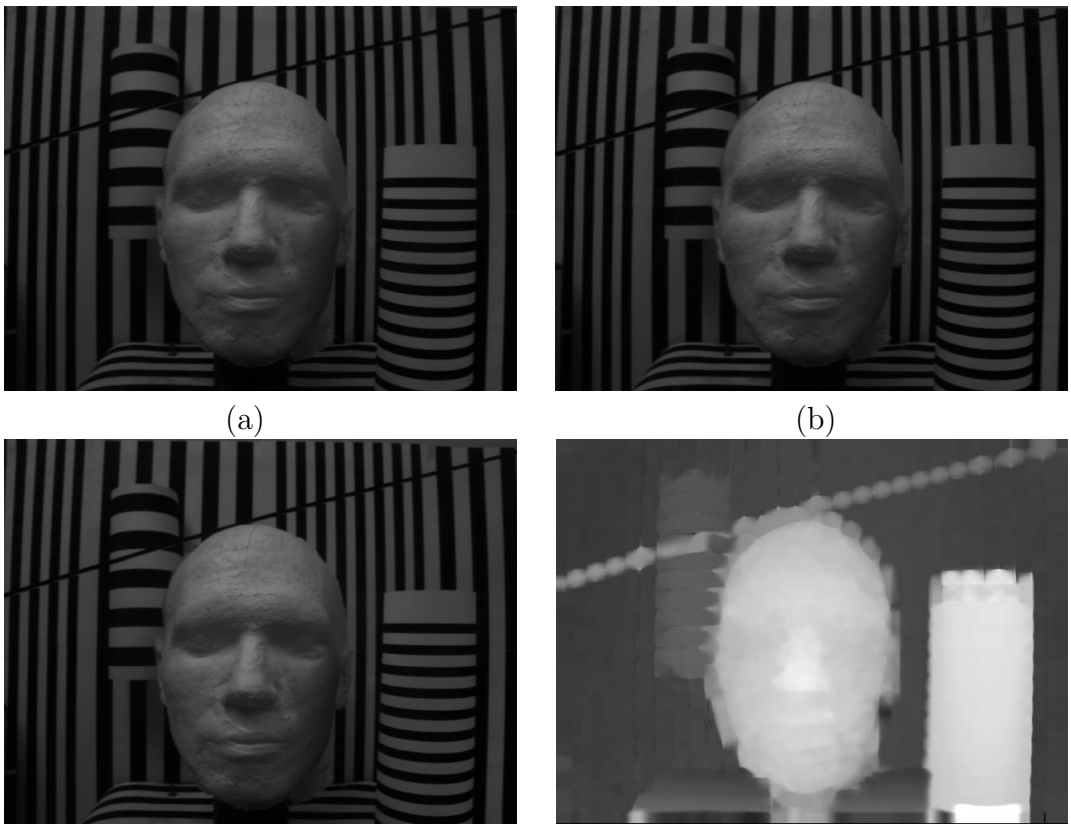
3.1.1 Related Work

Correspondence Based Methods

The standard approach to the problem of structure from motion is to first compute correspondences. These might be either dense correspondence in the form of optical flow [33, 43, 49] or feature correspondence [11, 83, 16]. Then the correspondences are used to compute the camera motion and scene structure. The advantage of our method over both optical flow methods [42, 43, 28, 55, 77] and feature based methods [79, 16, 65, 67, 4] is that no prior computation of correspondences is needed, a computation which in itself is error prone. Since we obtain a linear system of equations that combines the information from all the pixels in the image, we avoid the aperture problem. By avoiding the need to explicitly detect feature points we can use information from areas where gradient information is weak such as shadow edges.

These advantages are highlighted in a scene such as in Figure 3-1. The scene contains a background of straight bars, a plaster bust in the foreground, and an oblique line cutting the bars. The background bars provide an unreliable source of information for point-to-point correspondence because of the aperture problem. Even large regions of the image have no information which constrains the flow in the Y direction. Many feature points in the image do not correspond to real feature points in the 3D world. For example, the intersection of the bars and the line in the image do not correspond to physical features in 3D since the line does not lie on the background plane. Figure 3-2 shows a detail of the optical flow field in the upper left of the scene where these two problems occur. The flow was computed using a widely available, 'industrial strength', optical flow program [7, 6]. (For more examples see Section (4.2.3).)

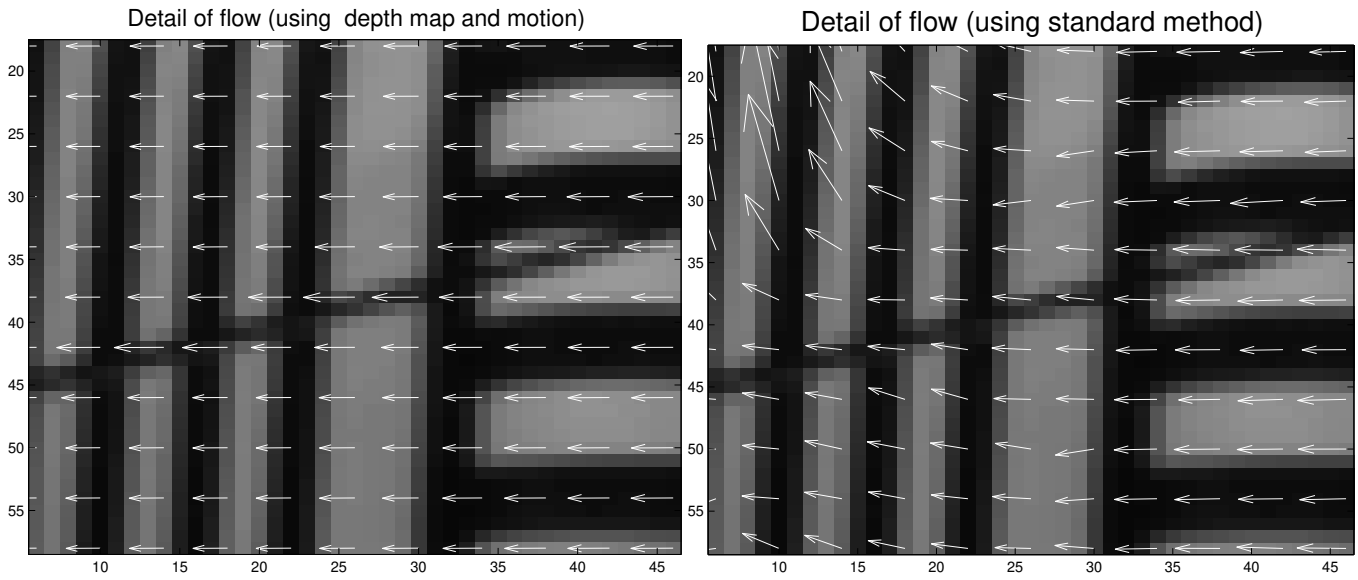
Finally, the texture on the plaster bust varies smoothly and is low contrast thereby giving rise only to a relatively small number of reliable features to track. Yet, our method produces a full 3D model of the scene as demonstrated in Figure 3-3*a,b*. Many natural scenes such as tree branches or man made objects such as window frames, lamp posts and fences, often give rise to these problems.



(a) (b)

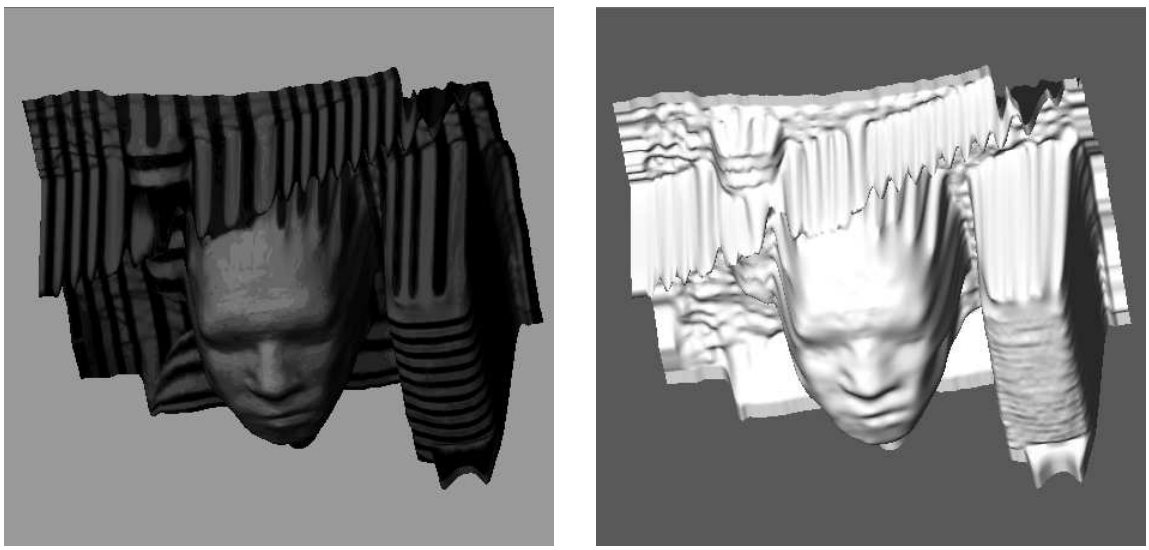
(c) (d)

Figure 3-1: *The three input images (a,b,c) and the estimated depth map (d). The motion between images (a) and (b) is horizontal. The motion is small but can be seen in the width of the rightmost vertical stripe and in the parallax between the cylinder on the right and the vertical stripes behind it. The motion of between images (a) and (c) was vertical.*



(a) (b)

Figure 3-2: *Detail of optical flow computed for Figure 3-1. (a) Correct optical flow computed using depth map and recovered motion. The camera motion was horizontal parallel to the image plane. The flow vectors have a zero Y component. (b) Optical flow computed using Bergen & Hingorani optical flow program. Note in the upper left corner the flow vectors have a strong Y component. This is due to the aperture problem where even a large window (aperture) will see image gradients in only one direction giving no constraint on the Y component of the flow. Near the intersections (in the image) of the diagonal line and the vertical bars the flow has a small Y component. This error occurs because the program 'tracks' the intersection point as if it were a real 'feature' point, but the lines do not intersect in space. Although smaller in magnitude this is the more significant error because the program will also give a high confidence to this value.*



(a) (b)

Figure 3-3: *3D rendering of the estimated depth map from images in Figure 3-1. These images show the inverse depth, $k = \frac{1}{z}$, which is the natural value to compute. In (b) the texture map was removed to show the detail and the flaws.*

Direct Methods

The 'direct methods' were pioneered by Horn & Weldon in [34]. Using only a single image pair they ended up with N equations in $N+5$ unknowns, where N is the number of points in the image. The unknowns are the N unknown depths and the 3 translation and 3 rotation parameters with one unknown dropping out because translation and depth can only be found up to a common scale factor. The problem is therefore ill posed and additional constraints are needed. Negahdaripour & Horn [52] present a closed form solution assuming a planar or quadratic surface. Szeliski & Kang [76] describe an iterative solution using splines to enforce a smoothness constraint on the depth. McQuirk [45] shows that in a pure translation model the subset of the image points with a nonzero spatial derivative but a zero time derivative gives the direction of motion, thus the *focus of expansion* (FOE) is on a line perpendicular to the gradient at these points. Since this uses only a subset of the image points, one ends up discarding most of the image data.

Heel [29] constructs a Kalman filter to build up a structure model from more than one image pair but the core computation is fundamentally the same single image pair computation. The basic idea is that the Horn & Weldon equation is linear in depth if motion is known and linear in motion if depth is known. Heel initially assumes a depth (constant depth surface) and estimates the motion and then uses the motion to estimate the depth. The depth map is then warped using the computed motion and used as the initial depth estimate for the next image pair in the sequence. The question of convergence is not answered and failures are not reported. Results are only shown for pure translation. Heel limits the motion to image motions of less than 3 pixels which is one of the limiting factors in the accuracy of the surface

reconstructions. A coarse-to-fine framework (Section 3.7.4) enables us to use much larger image motions, thus increasing the dynamic range by an order of magnitude.

Michaels [46] uses three frames (two motions). Each motion gives N equations totaling $2N$ equations. The unknowns are the N unknown depths and the 2×6 translation and rotation parameters. Again, one parameter drops out because of the translation and depth scale ambiguity. He solves the $2N$ equations with $N + 11$ unknowns as a large nonlinear optimization problem using the Levenberg-Marquart algorithm [47]. Since it took a very long time to converge results are shown for very low resolution images only. He also shows theoretically that a large field of view ($\approx 120^o$) is required for accurate direct estimation of motion given structure. Since estimation of motion given structure is a key stage in the process he suggests that a wide field of view is required for the whole process of estimation of structure and motion. Kumar et al [40] first align a dominant plane in the images. The residual image motion is epipolar motion (i.e. pure translation). The epipole of the residual motion is found using iterative techniques starting with an initial guess. Others who have explored planar alignment include [36, 56]. The theoretical framework for planar alignment can also be found in [60].

More recently, Fermuller & Aloimonos [18] describe global geometric properties of the flow-field that give rise to direct relationships between the measurement of normal flow and the ego-motion parameters solved by means of search techniques.

The tensor brightness constraint was first presented in [64] and a practical implementation with results was first described in [74]. During the course of implementation various subtleties and limitations were discovered.

3.1.2 Overview of this Chapter

In Section 3.2 we derive the the tensor brightness constraint, the model-based brightness constraint for the projective camera model. The constraint provides one linear equation per pixel in the 27 unknown elements of the trilinear tensor. This results in a highly over-constrained set of equations but we show that these equations are linearly dependent. A unique solution is obtainable by applying further admissibility constraints as shown in Chapter 6 but this added complexity in finding the unique solution leads us to search for simpler models.

We next consider the case of a calibrated camera with the rotation limited to small rotation angles (Section 3.3). This results in a 20-parameter model but again the linear solution is degenerate. Therefore we develop a simpler model in Section 3.4 using the small motion approximations of Longuet-Higgins & Prazdny [42]. This results in a 15-parameter model-based brightness constraint. Before showing how to solve for the ego-motion parameters for the general case of rotation and translation, we show how to solve for the motion parameters in the simpler example of pure translation in Section 3.5. This example is useful because it shows the basic method we employ to solve the linear homogeneous equations. In Section 3.6 we show how to solve the 15-parameter model for both translation and rotation.

In Section 3.7 we discuss the implementation issues and we provide the details needed to build a program that actually works.

3.2 The Tensor Brightness Constraint: the Projective Model

As we have seen in Chapter 2, a geometric trilinear matching constraint is produced by contracting the tensor with the point p of Image 1, *any* line coincident with p' in Image 2, and *any* line coincident with p'' in Image 3 (equation 2.41). In particular, we may use as the lines the tangents to the iso-brightness contours at p' and p'', respectively, as suggested by [69]. However, there still remains the problem of finding those matching tangents in the first place.

In order to avoid the need to first find correspondences we use the results from Section 2.5.2. The lines

$$s' = \begin{pmatrix} I_x \\ I_y \\ I'_t - xI_x - yI_y \end{pmatrix} \qquad s'' = \begin{pmatrix} I_x \\ I_y \\ I''_t - xI_x - yI_y \end{pmatrix} \tag{3.1}$$

are lines coincident with p' and p'' respectively and parallel to the iso brightness contour at (x, y). It is important to note that equations of the lines s' and s'' (equation 3.1) can be computed from the coordinates (x, y) and the derivatives I_x, I_y, I'_t, I''_t. I_x and I_y are the spatial derivatives in Image 1 at point (x, y). I'_t and I''_t are the temporal derivatives computed at the point (x, y). In practice $I'_t = I_2(x, y) - I_1(x, y)$ and $I''_t = I_3(x, y) - I_1(x, y)$. The coordinates of the corresponding points (x', y') and (x'', y'') are not required.

Substituting equations (3.1) into the tensor equation (2.41) results in the *tensor brightness constraint*:

$$\boxed{s''_k s'_j p^i \mathcal{T}_i^{jk} = 0.} \tag{3.2}$$

We have one such equation for each point on the image where s'_j and s''_k can be computed from the image gradients and $p = (x, y, 1)^T$ are the (projective) image coordinates of the point in Image 1. We wish to solve for $\mathcal{T}_i^{jk}$ the elements of which are the bilinear combinations of the motion and camera parameters.

Every pixel with a non-vanishing gradient contributes one linear equation to the 27 unknown parameters comprising $\mathcal{T}_i^{jk}$. However, a configuration of a point in the first image and two *parallel* lines in Images 2 and 3 is a particular instance of a degenerate line configuration called a Linear Line Complex. The general solution for the LLC case is explored in Chapter 6. In this particular case the system of equations can provide a linear solution to 21 of the parameters and the remaining 6 parameters of $\mathcal{T}_i^{jk}$ can be determined using quadratic admissibility constraints. We provide here a proof for this special case:

From equation (3.2) the coefficients of the terms $\mathcal{T}_i^{12}$ and the terms $\mathcal{T}_i^{21}$ are both $p^i I_x I_y$. Therefore the linear equations cannot be solved for the six terms ($\mathcal{T}_i^{12}$ and $\mathcal{T}_i^{21}$) individually but only for the three sums $\mathcal{T}_i^{12} + \mathcal{T}_i^{21}, i = 1...3$.

The sum $\mathcal{T}_i^{12} + \mathcal{T}_i^{21}$ provides one linear equation in $\mathcal{T}_i^{12}$ and $\mathcal{T}_i^{21}$. We will use the admissibility constraint on the Standard Correlation Slices, T_i. The constraint states

that the matrix T_i is of $rank = 2$. This leads to a quadratic equation in $\mathcal{T}_i^{12}$ and $\mathcal{T}_i^{21}$. The quadratic and linear equations together result in two solutions for each pair $\mathcal{T}_i^{12}, \mathcal{T}_i^{21}, i = 1...3$ for a total of 8 discrete solutions. A unique solution is obtainable by applying further admissibility constraints as shown in Chapter 6.

This added complexity in finding the unique solution leads us to search for simpler models. We next consider the case of a calibrated camera with the rotation limited to small rotation angles.

3.3 The Small Rotation Model with Calibrated Cameras

The next model is defined for small-angle rotations with calibrated cameras. Assuming calibrated cameras equations (2.36) become:

$$\begin{array}{rcccl} p' & \cong & [\mathbf{R}'; t']\, P & \cong & \mathbf{R}'p + kt' \\ p'' & \cong & [\mathbf{R}''; t'']\, P & \cong & \mathbf{R}''p + kt'' \end{array} \tag{3.3}$$

i.e., $\tilde{A} = [\mathbf{R}'; t']$ and $\tilde{B} = [\mathbf{R}''; t'']$ and $k = \frac{1}{z}$ replaces ρ in equation (2.36).

If we also assume small angle rotations and can therefore make the approximations:

$$\cos(\theta) \approx 1, \quad \sin(\theta) \approx \theta$$

then:

$$\begin{array}{rcl} p' & \cong & [I + [w']_x; t']\, P \\ p'' & \cong & [I + [w'']_x; t'']\, P \end{array} \tag{3.4}$$

where w', w'' are the angular velocity vectors and $[\cdot]_x$ is the skew-symmetric matrix of vector products. (i.e., $\tilde{A} = [I + [w']_x; t']$ and $\tilde{B} = [I + [w'']_x; t'']$).

We can now proceed along two paths. The different results will give us some further insight into the problem. First we will substitute $\tilde{A} = [I + [w']_x; t']$ and $\tilde{B} = [I + [w'']_x; t'']$ directly into equation (2.41) and then write out the tensor explicitly:

$$T_1 = \begin{bmatrix} t'_1 - t''_1 & t'_1 w''_3 - t''_2 & -t'_1 w''_2 - t''_3 \\ t'_2 - t''_1 w'_3 & t'_2 w''_3 - t''_2 w'_3 & -t'_2 w''_2 - t''_3 w'_3 \\ t'_3 + t''_1 w'_2 & t'_3 w''_3 + t''_2 w'_2 & -t'_3 w''_2 + t''_3 w'_2 \end{bmatrix} \tag{3.5}$$

$$T_2 = \begin{bmatrix} -t'_1 w''_3 + t''_1 w'_3 & t'_1 + t''_2 w'_3 & t'_1 w''_1 + t''_3 w'_3 \\ -t'_2 w''_3 - t''_1 & t'_2 - t''_2 & t'_2 w''_1 - t''_3 \\ -t'_3 w''_3 - t''_1 w'_1 & t'_3 - t''_2 w'_1 & t'_3 w''_1 - t''_3 w'_1 \end{bmatrix} \tag{3.6}$$

$$T_3 = \begin{bmatrix} t'_1 w''_2 - t''_1 w'_2 & -t'_1 w''_1 - t''_2 w'_2 & t'_1 - t''_3 w'_2 \\ t'_2 w''_2 + t''_1 w'_1 & -t'_2 w''_1 + t''_2 w'_1 & t'_2 + t''_3 w'_1 \\ t'_3 w''_2 - t''_1 & -t'_3 w''_1 - t''_2 & t'_3 - t''_3 \end{bmatrix} \tag{3.7}$$

where T_1, T_2 and T_3 are the Standard Correlation Slices (see Section 2.4.1). Studying the tensor terms we notice that the terms are not linearly independent and the

following seven linear equations hold:

$$\begin{array}{l}
\mathcal{T}_1^{1,2} + \mathcal{T}_2^{1,1} + \mathcal{T}_1^{2,1} - \mathcal{T}_2^{2,2} = 0 \\
\mathcal{T}_2^{2,1} + \mathcal{T}_1^{2,2} + \mathcal{T}_2^{1,2} - \mathcal{T}_1^{1,1} = 0 \\
\mathcal{T}_3^{1,1} + \mathcal{T}_1^{1,3} + \mathcal{T}_1^{3,1} - \mathcal{T}_3^{3,3} = 0 \\
\mathcal{T}_1^{3,3} + \mathcal{T}_3^{3,1} + \mathcal{T}_3^{1,3} - \mathcal{T}_1^{1,1} = 0 \\
\mathcal{T}_2^{3,3} + \mathcal{T}_3^{3,2} + \mathcal{T}_3^{2,3} - \mathcal{T}_2^{2,2} = 0 \\
\mathcal{T}_3^{2,2} + \mathcal{T}_2^{2,3} + \mathcal{T}_2^{3,2} - \mathcal{T}_3^{3,3} = 0 \\
\mathcal{T}_2^{1,3} + \mathcal{T}_3^{2,1} + \mathcal{T}_1^{2,3} + \mathcal{T}_3^{1,2} + \mathcal{T}_1^{3,2} + \mathcal{T}_2^{3,1} = 0
\end{array} \tag{3.8}$$

Therefore, for the calibrated small rotation model, the 27-parameter Tensor Brightness Constraint reduces to a *20-parameter* constraint equation. Each point in the image gives one homogeneous equation resulting in a set of N homogeneous equation, where N is the number of image points. This can be written in matrix form as:

$$\mathbf{A}x = 0$$

where $\mathbf{A}$ is the $N \times 20$ estimation matrix and x is the 20 parameter vector of the unknown intermediate parameters. There is a unique non-trivial solution if the estimation matrix is of $rank = 19$. But as in the projective case, the system of homogeneous equations is degenerate and in this case the rank of the estimation matrix is 16 not 19.

In [64, 74] a model-based brightness constraint for the calibrated, small rotation model was derived in a different way. Expanding equation (3.4):

$$p' \;\cong\; [I + [w']_x]\, p + kt' \tag{3.9}$$

and taking the dot product of the left-hand-side and right-hand-side terms with s' (equations 3.1) results in the equation:

$$0 \;=\; s'^{\top} p + s'^{\top} [w']_x + k s'^{\top} t' \tag{3.10}$$

This can be simplified to the form:

$$k s'^{\top} t' + v'^{\top} w' + I'_t \;=\; 0 \tag{3.11}$$

where:

$$v' = p \times s' = \begin{pmatrix} -I_y + y(I'_t - xI_x - yI_y) \\ I_x - x(I'_t - xI_x - yI_y) \\ xI_y - yI_x \end{pmatrix}.$$

In a similar manner, for the second motion:

$$k s''^{\top} t'' + v''^{\top} w'' + I''_t \;=\; 0 \tag{3.12}$$

where

$$v'' = p \times s'' = \begin{pmatrix} -I_y + y(I''_t - xI_x - yI_y) \\ I_x - x(I''_t - xI_x - yI_y) \\ xI_y - yI_x \end{pmatrix}.$$

Multiplying equations (3.11) by $s''^{\top}t''$ and equations (3.12) by $s'^{\top}t'$ and subtracting we obtain the following *24-parameter* model-based brightness constraint as a reduction of the tensor brightness constraint:

$$I''_t s'^{\top}t' - I'_t s''^{\top}t'' + s'^{\top}[t'w''^{\top}]v'' - s''^{\top}[t''w'^{\top}]v' = 0. \tag{3.13}$$

The resulting 24-parameter model (equation 3.13) is not as compact as the 20-parameter model we obtained by direct substitution of the small rotation approximation into the tensor brightness constraint equation. We find that the $24 \times N$ estimation matrix obtained using the 24-parameter model is again of rank=16. Due to the high degeneracy of the linear equations neither the 24-parameter model nor the 20-parameter model are convenient to work with.

3.4 The 15-Parameter, Small Motion Model

A unique solution is obtained when we reduce the motion model further to include infinitesimal motion using the model introduced by Longuet-Higgins & Prazdny [42] (see Section 2.2.4). This model assumes in addition to small rotation that $\frac{t'_z}{Z} \ll 1$. The motion field equations for the first camera motion are then:

$$\begin{aligned} u' &= \frac{1}{z}(t'_1 - xt'_3) - w'_3 y + w'_2(1 + x^2) - w'_1 xy \\ v' &= \frac{1}{z}(t'_2 - yt'_3) + w'_3 x - w'_1(1 + y^2) + w'_2 xy \end{aligned} \tag{3.14}$$

By substituting equation (3.14) into the optical flow constraint equation (equation2.53, Section 2.5):

$$u'I_x + v'I_y + I'_t = 0 \tag{3.15}$$

and rearranging the terms we obtain:

$$ks^{\top}t' + v^{\top}w' + I'_t = 0 \tag{3.16}$$

where $k = \frac{1}{z}$ denotes the inverse depth at each pixel location and where s, v are defined below:

$$s = \begin{pmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{pmatrix} \tag{3.17}$$

and

$$v = p \times s = \begin{pmatrix} -I_y - y(xI_x + yI_y) \\ I_x + x(xI_x + yI_y) \\ xI_y - yI_x \end{pmatrix} \tag{3.18}$$

Note that v is a vector that can be computed from image brightness, and should not be confused with v' which is the y component of the image flow between Image 1 and Image 2. (Equation (3.16) which was first derived in [52, 34] can also be derived

directly from equation (3.11) by making the LH&P assumptions: $\frac{t'_z}{Z} \ll 1$, $\frac{w'_y x}{f} \ll 1$ and $\frac{w'_x y}{f} \ll 1$.)

Similarly for the second motion:

$$ks^\top t'' + v^\top w'' + I''_t = 0 \tag{3.19}$$

By eliminating k from equations (3.16) and (3.19) we obtain the *15-parameter model-based brightness constraint*:

$$\boxed{I''_t s^T t' - I'_t s^T t'' + s^T [t' w''^T - t'' w'^T] v = 0} \tag{3.20}$$

Equation (3.20) is a key equation. There is one such equation for every point in the image. The unknowns are the motion parameters t', t'', w' and w''. The values I'_t, I''_t, s and v, can all be computed from image derivatives and the coordinates of the point in Image 1. Given the solution to the ego-motion parameters (to be described later), one can recover the dense depth map from equations (3.16) and (3.19).

Before showing how to solve equation (3.20) for both rotation and translation we will start with the simple case of pure translation. This simple will be used to explain the basic method we employ to find the motion parameters. The pure translation case arises in practice when we can either assure pure translation or we have previously rectified the images (perhaps by registering a common plane in the images as in Section 3.6.4). It is also an important case from a theoretical point of view. We will show that there are problems in recovering the motion parameters in the case of collinear motion. If collinear motion causes a problem for the pure translation case, it will also create a problem for the more general translation and rotation case.

3.5 Pure Translation

The pure translational small-motion model takes the simplest form:

$$I''_t s^T t' - I'_t s^T t'' = 0 \tag{3.21}$$

We have one such equation for each image point and we can write it out in the matrix form:

$$\mathbf{A}t = 0$$

where:

$$t = \begin{pmatrix} t'_x & t'_y & t'_z & t''_x & t''_y & t''_z \end{pmatrix}^T$$

and $\mathbf{A}$ is an $N \times 6$ matrix with the $i'th$ row (corresponding to the $i'th$ pixel) given by:

$$\begin{pmatrix} I''_t s_{i1} & I''_t s_{i2} & I''_t s_{i3} & I'_t s_{i1} & I'_t s_{i2} & I'_t s_{i3} \end{pmatrix}$$

To avoid the trivial solution $t = 0$ we add the constraint $\|t\| = 1$. The least squares problem now maps to the problem of finding $\|t\| = 1$ that minimizes:

$$t^T \mathbf{A}^T \mathbf{A} t = 0$$

The solution is the eigenvector of $\mathbf{A}^T\mathbf{A}$ corresponding to the smallest eigenvalue.

3.5.1 The Singular Case of Collinear Motion

This method fails when the two motions are in the same (or opposite) directions. This is obvious in the pure translation case where:

$$I''_t s^\top t' - I'_t s^\top t'' = 0 \tag{3.22}$$

and

$$k s^\top t' + I'_t = 0 \tag{3.23}$$
$$k s^\top t'' + I''_t = 0 \tag{3.24}$$

If the second translation vector t'' is proportional to the translation vector t' then equation (3.24) is simply a scaled version of equation (3.23) adding no new information. The solution will therefore be ill conditioned when $t' = \beta t''$ for a scalar β. This is a drawback in many applications (e.g. 3D reconstruction from a monocular image sequence). We present in Section 3.8.1 some early research on ways to overcome this problem. The initial results look promising.

3.6 Solving for Translation and Rotation

In equation (3.20) the ego-motion parameters are embedded in a 15 parameter model: 6 translation parameters and the 9 outer-product terms $[t'w''^T - t''w'^T]$. Each pixel provides one linear equation, thus N pixels provide a system $\mathbf{A}c = 0$ where $\mathbf{A}$ is an $N \times 15$ measurement matrix and c is the 15-vector of unknown parameters. The least squares solution is the eigenvector of $\mathbf{A}^\top\mathbf{A}$ corresponding to the smallest eigenvalue. Without noise the eigenvalue will be zero and the solution is in the null space of $\mathbf{A}^\top\mathbf{A}$. The solution is unique if the null space is one dimensional and the rank of $\mathbf{A}$ is 14. We will prove however, that the null space is at least two dimensional.

Proposition 5 *The rank of* $\mathbf{A}$ *is 13 not 14*

Let $\mathbf{A}$ *be the* $N \times 15$ *measurement matrix associated with the homogeneous equation (3.20), where* $N \gg 15$*. Then the null space of the matrix* $\mathbf{A}^\top\mathbf{A}$ *is of dimension greater or equal to 2 and* $Rank(\mathbf{A}) \leq 13$*.*

Proof:
Since

$$s^\top v = s^\top (p \times s) = 0$$

the vector

$$c_0 = (0,0,0,0,0,0,1,0,0,0,1,0,0,0,1)^T$$

is in the null space of $\mathbf{A}$.
The vector c_0 corresponds to:

$$t' = 0$$
$$t'' = 0$$
$$\left[t'w''^T - t''w'^T\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore c_0 is clearly not an admissible solution, and therefore the null space of $\mathbf{A}$ includes two vectors: the vector c_0 and the true solution. It follows that the rank of $\mathbf{A}$ is at most 13. □

The method of solution is based on the following observation. Let b_0 be another vector of the null-space (say c_0 and b_0 are the two eigenvectors of $\mathbf{A}^\top\mathbf{A}$ corresponding to the two smallest eigenvalues), then the desired solution vector b is a linear combination of the two:

$$b = b_0 + \alpha c_0 \tag{3.25}$$

We note that c_0 is in the null space of $\mathbf{A}$ regardless of noise in the data and will therefore correspond to an eigenvalue of zero (up to numerical precision). Theoretically, b_0 is also in the null space of $\mathbf{A}$ but only in the case of noise free data and exact motion model. Therefore b_0 is the eigenvector corresponding to the second smallest eigenvalue. In order to find α given c_0 and b_0 we enforce the constraint that $Rank[t'w''^T - t''w'^T] = 2$.

3.6.1 Solution Using the Rank=2 Constraint.

Clearly the choice of α will have no affect the first 6 elements of the vector b. Let us arrange the last 9 elements of b, b_0 and c_0 into the corresponding 3×3 matrices $\mathbf{B}$, $\mathbf{B_0}$ and $\mathbf{C_0}$. We are now looking for an α such that:

$$Rank(\mathbf{B_0} - \alpha\mathbf{C_0}) = 2. \tag{3.26}$$

In our case $\mathbf{C_0}$ is the identity matrix so the solution for α is given simply by the eigenvalues of $\mathbf{B_0}$. Since $\mathbf{B_0}$ is a 3×3 matrix this results in up to three discrete solutions. We now prove that only one of those can be a valid solution.

Theorem 1 *The 15-parameter model has a unique solution*
Let:

$$\mathbf{B} = \left[t'w''^T - t''w'^T\right]. \tag{3.27}$$

Let:

$$\tilde{\mathbf{B}} = \left[t'w''^T - t''w'^T\right] + \alpha\mathbf{I}. \tag{3.28}$$

Then the equation:

$$\tilde{\mathbf{B}} = \left[t'\tilde{w}''^T - t''\tilde{w}'^T\right] \tag{3.29}$$

has a solution for $\tilde{w}'', \tilde{w}'$ given t', t'' iff $\alpha = 0$.

Proof:

The *if* part of the proof is trivial. For $\alpha = 0$, $\tilde{w}'' = w''$ and $\tilde{w}' = w'$ is a solution for (3.29). To prove the *only if* part, suppose there exist $\tilde{w}'', \tilde{w}'$ that is a solution for (3.29). Then:

$$\left[t'w''^T - t''w'^T\right] + \alpha\mathbf{I} = \left[t'\tilde{w}''^T - t''\tilde{w}'^T\right]. \tag{3.30}$$

Let q be a vector such that $q^Tt' = 0$, $q^Tt'' = 0$ and $q \neq \vec{0}$. If t' and t'' are not parallel we can choose:

$$q = t' \times t''$$

Multiply left and right side of (3.30) by q^T to get:

$$q^T \left[t' w''^T\right] - q^T \left[t'' w'^T\right] + q^T \alpha \mathbf{I} = q^T \left[t' \tilde{w}''^T\right] - q^T \left[t'' \tilde{w}'^T\right]. \tag{3.31}$$

Since $q^T t' = 0$ and $q^T t'' = 0$ most of terms in (3.31) become zero and we are left with:

$$q^T \alpha \mathbf{I} = \vec{0}. \tag{3.32}$$

Since $q \neq \vec{0}$ then $\alpha = 0$. □

3.6.2 The Algorithm for Finding Motion Parameters

Based on the previous arguments the algorithm for finding the motion parameters is as follows:

1. Compute the $N \times 15$ matrix $\mathbf{A}$ from equation (3.20).

2. Find the eigenvector corresponding to the second smallest eigenvalue of the matrix $\mathbf{A}^T\mathbf{A}$. This is b_0. (The vector c_0 corresponds to the smallest.)

3. The first 6 elements of b_0 are the translations, t',t''.

4. Arrange the last 9 elements of b_0 into a 3×3 matrix $\mathbf{B_0}$.

5. Construct three possible rank 2 matrices $\mathbf{B}_{i,i=1..3}$ from $\mathbf{B}_0$:

$$\mathbf{B}_i = \mathbf{B}_0 - \alpha_i \mathbf{I} \tag{3.33}$$

   where $\alpha_{i,i=1..3}$ are the three eigenvalues of $\mathbf{B}_0$.

6. For each $\alpha_{i,i=1..3}$ solve:

$$[t' w''^T - t'' w'^T] = \mathbf{B},$$

   for w' and w'', given t' and t'' from step (3). This a set of 9 linear equations in the six unknowns (w',w'') and is solved using least squares.

7. From Theorem 1 only one of the three solutions is correct. For each of the three solutions use the 12 parameters t', t'', w' and w'' to form the 15 intermediate parameter vector b_i and then compute the error:

$$E_i = b_i^T \mathbf{A}^T \mathbf{A} b_i. \tag{3.34}$$

   Of the three solutions select the solution which gives the smallest error.

3.6.3 Solve as a Nonlinear Optimization

In Section 3.6.2 we use linear methods to solve for the motion parameters. In order to do so we have treated the 15 intermediate parameters as linearly independent parameters while they are in fact bilinear combinations of 12 independent parameters. It is often possible to improve the motion estimates by using the linear solution as a starting point for non-linear optimization techniques [15]. In our experiments we did not find that a nonlinear optimization stage improved the results. This is possibly because the iterative framework (Section 3.7.2) is in itself a form of nonlinear optimization.

The nonlinear optimization procedure is brought here for completeness. Our general problem is to find motion parameters t', t'', w', w'' that minimize the cost function:

$$E(t', t'', w', w'') = b^\top \mathbf{A}^\top \mathbf{A} b \tag{3.35}$$

where $\mathbf{A}$ is the estimation matrix derived from equation (3.20) and $b(t', t'', w', w'')$ is the fifteen element vector of bilinear functions of the motion parameters.

One method would be to find motion parameters t', t'', w', w'' that minimize equation (3.35). In order to avoid the trivial solution we need to add a constraint such as: $|t'|^2 + |t''|^2 = 1$. This becomes a 12 parameter nonlinear constrained optimization problem.

The process can be simplified by noting that equation (3.20) can be rewritten in the form:

$$\left(I_t'' - w''^\top v''\right) s'^\top t' - \left(I_t' - w'^\top v'\right) s''^\top t'' = 0. \tag{3.36}$$

given a set of rotation values w' and w'', one can compute the least squares estimate of t' and t'' using linear methods and also compute the least squares error (equation 3.35). One can then use nonlinear search techniques to find w' and w'' which minimize this least squares error. This becomes a six parameter unconstrained optimization problem.

3.6.4 Reduce to Pure Translation by Planar Alignment

As an alternative to solving the set of bilinear equations we can follow the ideas suggested by [36, 56, 40] (see also [60]) and first align a dominant plane in the image. After planar alignment the resulting image motion is due to pure translation, and we can use pure translation version of the tensor brightness constraint.

The resulting 3D geometry will be relative to the aligned plane. Therefore it is critical that the same image region (planar surface) is used to align both image motions. The resulting reconstruction will be a projective transformation of the true Euclidean shape and the motion field information will be correct. For some purposes such as finding correspondences or depth discontinuities this is sufficient. If needed the true (Euclidean) translation and rotation can be found as a second step [36].

3.7 Implementation Details

In this section we will provide the details needed to build a program that actually works.

3.7.1 Computing the Depth, Smoothing and Interpolation

After recovering the camera motion (Section 3.6.2) we use equations (3.16) and (3.19) to compute depth at every point. Information is combined from both image pairs by minimizing the least squares error:

$$E = \min_k \arg \sum_{j=1}^{2} \left(k s^T t^j + v^T w^j + I_t^j \right)^2 .$$

Here $j = 1$ and $j = 2$ denote values from the first and second image pairs respectively.

There are points in the image where the brightness gradients are close to zero, (and therefore $s^T t^j \simeq 0$) and the estimation of k will be ill-conditioned at those points. In order to overcome this problem we use a local region of support around the point of interest. It is simplest form we assume the depth is constant in the region and minimize:

$$E = \min_k \arg \sum_{x,y \in R} \sum_j \beta(x,y) |s^T t^j|^p \left(k s^T t^j + v^T w^j + I_t^j \right)^2 \tag{3.37}$$

where the windowing function $\beta(x, y)$ allows one to increase the weights of the points closer to the center of the region. The $|s^T t^j|^p$ term reduces the weight of points which have a small gradient or where the gradient is perpendicular to the camera motion since these cases are highly affected by noise. We used $p = 1$.

During the iteration process we typically used a region R of 7×7 to 11×11. For 'prettier' results the last iteration, we typically reduced the region R to 1×1 but added a very weak global smoothness term and performed multi-grid membrane interpolation[53]. The smoothness term stabilizes regions where there is no image gradient so very small regions of support can be used.

3.7.2 Iterative Refinement

The constant brightness constraint is a linearized form of the Sum Square Difference (SSD) criteria. The linear solution can be thought of as a single iteration of Newton's method applied to the problem. Iterative refinement is performed as follows:

1. Calculate motion (using equation 3.20).
2. Compute depth (using equation 3.37).
3. Using the depth and motion, warp Images 2 and 3 towards Image 1 (Section 3.7.3).

4. Compute new time derivatives I'_t and I''_t.

5. Compute a new motion and depth estimate.

In the ideal case, as the final result, the warped images should appear nearly identical to Image 1. One must be careful and not simply compute the incremental changes in the translation $\delta t'$ and $\delta t''$. As the images are warped closer together and the translation estimate approaches zero, the system of equations (3.20) will become ill conditioned. Furthermore, since the equations are homogeneous we must enforce a constraint such as $|t'|^2 + |t''|^2 = 1$ to avoid the trivial solution. We would not wish to apply such a constraint to $\delta t'$ and $\delta t''$. Thus, one must compute the full translation model (previous iteration plus the incremental change to the translation). This problem does not arise for the rotations: w' and w''. It is in fact convenient to warp the images using the best rotation estimate and then compute only the incremental improvement in the rotations: $\delta w'$ and $\delta w''$. For example we might chose to warp the images using a more exact rotation model than the linearized small rotation model.

The method for computing the full translation model and incremental rotation is now described. Let Ψ_1, Ψ_2 and Ψ_3 be the three images. Assume we have $\hat{k}$, $\hat{t}^j$, $\hat{w}^j$, from the previous iteration. (Here we use j to stand for either $'$ or $''$.) The translation components of image motion $\hat{u}_t^j$ and $\hat{v}_t^j$, and the rotational components $\hat{u}_r^j$ and $\hat{v}_r^j$ can be computed using $\hat{k}$, $\hat{t}^j$ and $\hat{w}^j$ in equation (3.14). These are then used to warp images Ψ_2 to $\hat{\Psi}_2$ and Ψ_3 to $\hat{\Psi}_3$. After warping, the images satisfy the brightness constraint equation:

$$\begin{aligned} I_x du' + I_y dv' + \hat{I}'_t = 0 \\ I_x du'' + I_y dv'' + \hat{I}''_t = 0 \end{aligned} \tag{3.38}$$

where the temporal derivatives at each pixel are given by:

$$\begin{aligned} \hat{I}'_t = \hat{\Psi}_2 - \Psi_1 \\ \hat{I}''_t = \hat{\Psi}_3 - \Psi_1 \end{aligned} \tag{3.39}$$

and du^j, dv^j are the (still unknown) differences between computed image motions and the real image motions:

$$\begin{aligned} du^j = u^j - \hat{u}_t^j - \hat{u}_r^j \\ dv^j = v^j - \hat{v}_t^j - \hat{v}_r^j \end{aligned} \tag{3.40}$$

Let:

$$\alpha^j = I_x \hat{u}_t^j + I_y \hat{v}_t^j \tag{3.41}$$

which can also be written as:

$$\alpha^j = \hat{k} s^T \hat{t}^j \tag{3.42}$$

Substituting equations (3.41) and (3.40) in equation (3.38) we get:

$$\begin{aligned} I_x(u' - \hat{u}_r^{'}) + I_y(v' - \hat{v}_r^{'}) + (\hat{I}_t' - \alpha') = 0 \\ I_x(u'' - \hat{u}_r^{''}) + I_y(v'' - \hat{v}_r^{''}) + (\hat{I}_t'' - \alpha'') = 0 \end{aligned} \tag{3.43}$$

Substituting equation (3.14) in equation (3.43) we get modified versions of the equations (3.16) and (3.19)

$$\begin{aligned} ks^T t' + v^T \delta w' + (I_t' - \alpha') = 0 \\ ks^T t'' + v^T \delta w'' + (I_t'' - \alpha'') = 0 \end{aligned} \tag{3.44}$$

We start our first iteration with $\hat{k}$, $\hat{t}^j$, $\hat{w}^j$ all zero and therefore $\alpha = 0$ as well.

3.7.3 Image Warping

Given an estimate of the camera motion and the depth at every point, we can warp Image 2 towards Image 1. Image warping in general is described in [87]. Let I'_{old} be the original Image 2 and let I'_{new} be the warped image we are trying to create. We must first define functions $x_{old}(x, y, ...)$ and $y_{old}(x, y, ...)$ which, given image coordinates in the new image (x, y) and possibly some extra parameters (symbolized by , ...), return the coordinates of the corresponding point in the old image (x_{old}, y_{old}).

In our case the *extra parameters* are the camera motions t' and w', and the depth map at every point $k(x, y)$. The functions x_{old} and y_{old} depend on the motion model. In our implementation we used the LH&P small motion model (equations (3.14)) but we could have also used the exact motion for calibrated cameras given in equations (2.15). Therefore:

$$x_{old} = x + u' = x + \tfrac{1}{z}(t'_1 - xt'_3) - w'_3 y + w'_2(1 + x^2) - w'_1 xy \tag{3.45}$$

and

$$y_{old} = y + v' = y + \tfrac{1}{z}(t'_2 - yt'_3) + w'_3 x - w'_1(1 + y^2) + w'_2 xy. \tag{3.46}$$

After defining x_{old} and y_{old} we can compute the value for every pixel in I'_{new} according to the formula:

$$I'_{new}(x, y) = \begin{cases} I'_{old}(x_{old}(x, y, ...), y_{old}(x, y, ...)), & if (x_{old}, y_{old}) \in I'_{old} \\ 0, & otherwise. \end{cases} \tag{3.47}$$

There are two points to note:

- The values of x_{old} and y_{old} are in general non-integer, so we use bilinear interpolation to compute the appropriate pixel values. Bilinear interpolation works better than nearest neighbor. The use of the more complex bicubic interpolation caused no noticeable improvement over bilinear interpolation.

- If the coordinates (x_{old},y_{old}) are outside the coordinates of the image then we use the value 0. It would make sense to mark these pixels as invalid and not use them in future motion and depth computation. This has not been implemented.

3.7.4 Coarse-to-Fine Processing

In order to deal with image motions larger than 1 pixel we use a Gaussian pyramid for coarse to fine processing [5][10]. Each of the three images is filtered by an approximation to a Gaussian filter and subsampled to create an image of half the size (in each dimension). This operation is performed recursively to create a pyramid with four or five levels depending on the size of the original image. The following 5×5 kernel was used as the filter:

$$\frac{1}{256} \times \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

This filter is separable and can be implemented by convolving the rows and columns of the image, twice each, with the filter $f = (0.25,\ 0.5,\ 0.25)$.

Starting from the coarsest level of we perform a few iterations to compute motion and depth according to the scheme described in Section 3.7.2. The brightness derivatives (I_x, I_y, I'_t, I''_t) are computed using the subsampled images. After computing the motion and depth, we create a finer depth map from the coarse depth map using bilinear interpolation. We use the interpolated depth map, and the motion estimates, as the starting values for the iterations at the finer level.

As we reduce the resolution of the image we must also scale the focal length accordingly (divide by 2). As we move to a finer level of the depth map, the depth values and motion estimates do not need scaling.

3.7.5 Field of View

In this section we will first demonstrate how in the general problem of determining camera motion from motion fields, when the field of view is narrow, there exists an ambiguity between rotation and translation. We will also note the errors in the estimation of the rotation if we use the wrong focal length value. Then we show how this ambiguity is observed in the shape of the cost function (equation 3.35) from three views. Finally we describe how to achieve stable and accurate motion estimates in the presence of rotation even without very wide fields of view.

Rotation-Translation Ambiguity from Motion Fields

It is well known that for a medium to narrow field of view, the motion field due to camera rotation around the Y axis, can be indistinguishable from the motion field due to some translation along the X axis and an appropriate depth surface. A similar ambiguity exists between rotations around the X axis and translations along the Y axis.

In Figure 3-4 we see that for medium-narrow fields of view (30^o) the flow due to rotation around the Y axis is parallel to the X axis. The length of the flow vectors due

to rotation is slightly shorter, closer to the Y axis. A similar flow field can be produced by camera translation along the X axis with a surface that is curved so that points in the center of the image are more distant. There are motion flows that unambiguously indicate a translational component to the motion: motions fields that include a focus-of-expansion or show parallax effects. Nevertheless even these fields are ambiguous since a rotational component could be added to the motion, compensated for by a change in the translation and depth to produce the same motion field. In some case the resulting depth map will include negative depths indicating some obvious error. This is often formulated as the *depth positive constraint* which can be used to limit the range of motion ambiguity.

With wider fields of view (i.e. shorter focal lengths) the ambiguity disappears. The motion fields due to rotation cannot be modeled by translational motion. Figure 3-4 also shows the motion fields due to rotation with fields of view of 53^o and 90^o. The distribution of the Y component of the flow has the unique characteristics of rotational flow. For wide fields of view the Y component points towards the X axis on the right of the image and away from the X axis on the left of the image (for this particular sign of the rotatation).

The motion field for a wide field of view is unambiguous if we know the focal length, or equivalently, the field of view (i.e. the camera's internal parameters are known). If our estimate of the focal length is incorrect then the motion and structure estimates which best fit the flow field will be incorrect. If the focal length estimate is smaller than the true focal length then estimate angle of rotation estimate will be smaller than the true rotation angle (in absolute values). Figure 3-5a shows the X and Y components of the motion field of a rotation of -4.5^o around the Y axis using a focal length of 400_{pixels}. Figure 3-5b shows the same measurements for a focal length of 200_{pixels} and a rotation of -2.5^o. The Y components are similar. The X components are very different but these can be accounted for by adjusting the translation and depth estimates.

Rotation-Translation Ambiguity in the 15-parameter Model

The rotation translation ambiguity can also be observed while using the 15-parameter model. Figure 3-6 show the shape of the cost function (equation 3.35) near the global minimum. These are simulation results for focal lengths of 50, 100 and 200. The image size was 320×240. It is not possible to plot the full 12 dimensional surface so the figures show only the shape of the function as we vary two motion parameters at a time. We show the effect of varying w'_x and t'_y where we expect to see ambiguity and the effect of varying w'_x and t'_x where no ambiguity is expected. In Figure 3-6f, the long diagonal valley shape of the cost function indicates the ambiguity of the motion estimate for narrower fields of view.

The Solution

As we have seen, in order to get reasonable rotation and translation estimates a wide field of view is required. The theoretical results of [46] and simulation results indicate

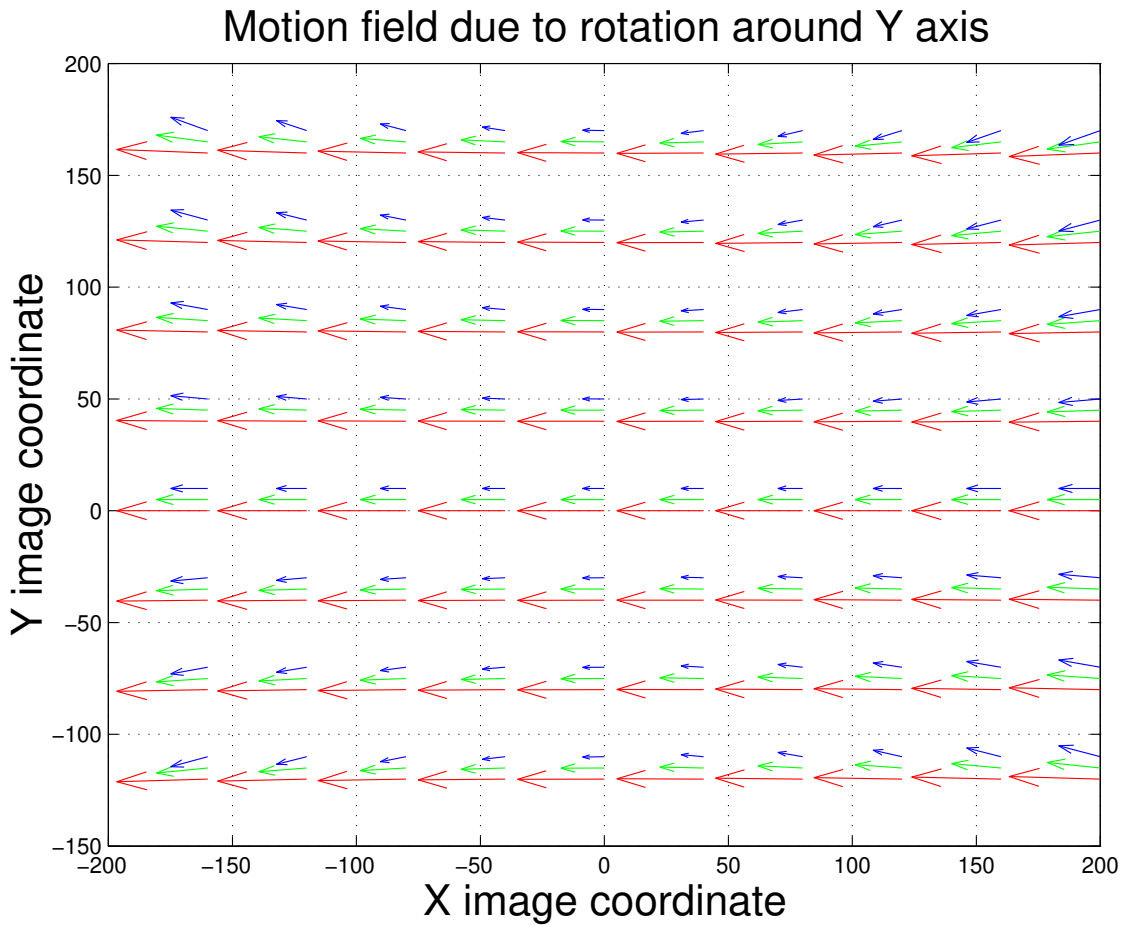
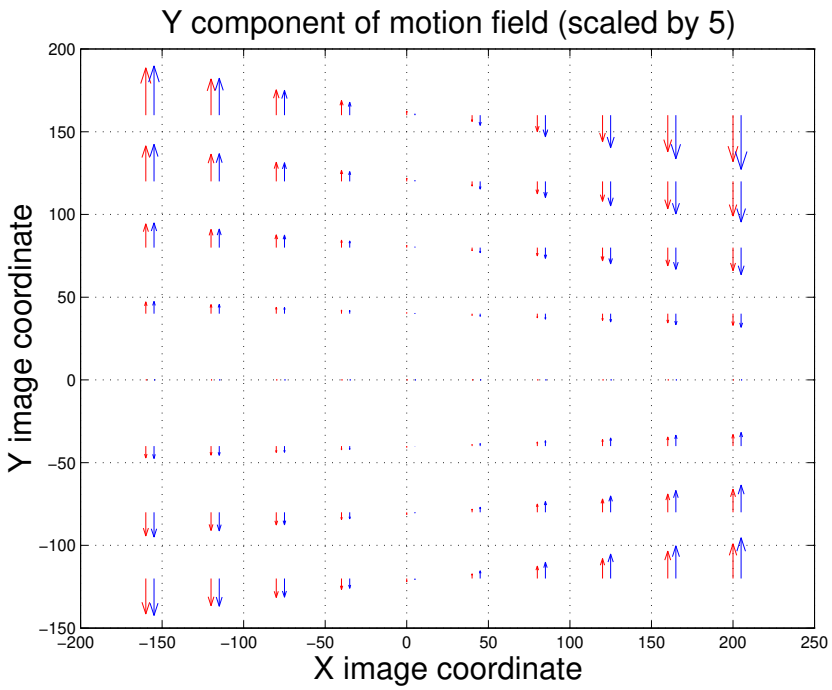
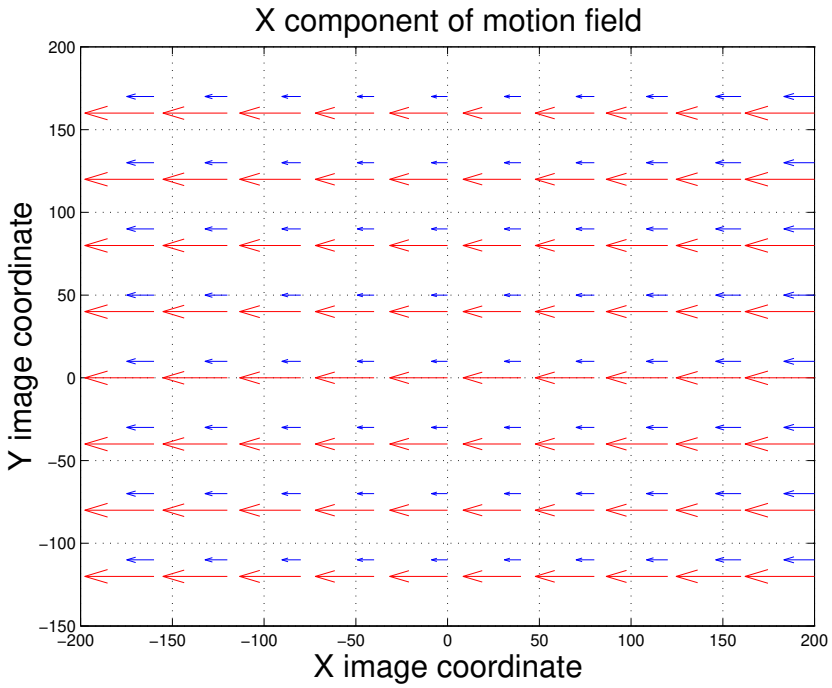


Figure 3-4: *Motion flow field due to rotation around the Y axis for wide, medium and narrow fields of view. The arrows for the medium and wide have been offset in the Y direction for clarity. For the narrow field of view ($f = 800_{pixels}$, FOV=30^o) the flow field is large and almost parallel to the X axis. For wider fields of view ($f = 400$ and $f = 200$) the X component of the flow is smaller but there is a noticeable Y component.*



(a)



(b)

Figure 3-5: *The X and Y components of the motion field due a rotation around the Y axis of -4.5^o with a focal length of 400_{pixels} for a rotation of -2.5^o with a focal length of 200_{pixels}. The arrows for the 200_{pixels} camera have been plotted offset right (a) and up (b). Note that the Y flow components are very similar.*

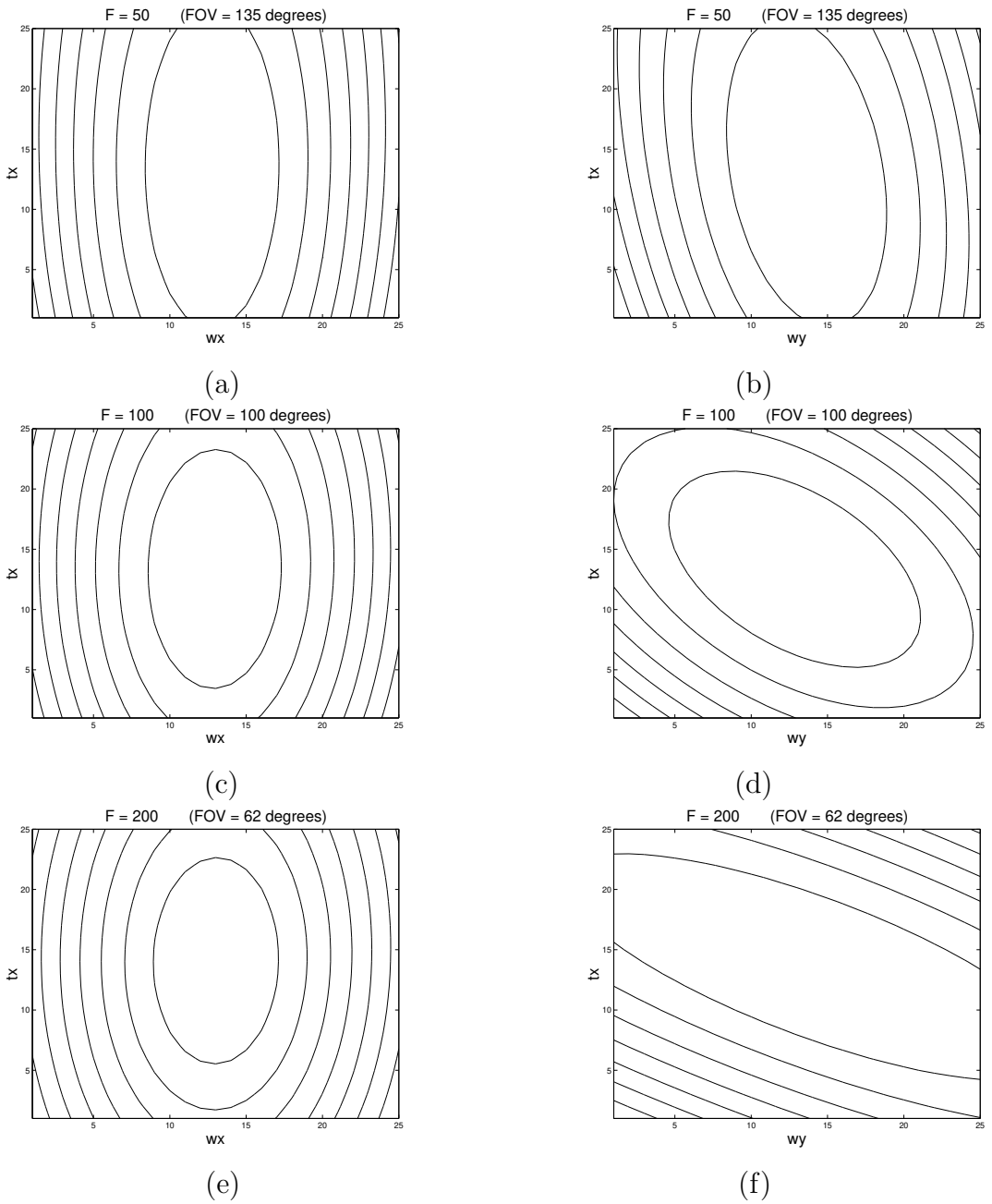


Figure 3-6: *The shape of the cost function around the minimum for various fields of view. These simulation results were computed after the program converge to the final motion and depth estimates. For visualization only two of the 12 parameters were changed at one time. Note that for a narrower field of view (f) the cost function has the shape of a narrow diagonal valley. This indicates an ambiguity between translation along the X axis and rotations around the Y axis.*

that very wide fields of view are required (120^o or greater). These are hard to achieve with standard lenses. With narrower fields of view the iterative scheme described in Section 3.7.2 fails to converge. In previously published work [74] we managed to stabilize the results for small rotation angles by unknowingly biasing the results towards small rotation. This was done by setting the focal length parameter to 50 instead of around 600. For a 640×480 size image this implied a field of view of 150^o instead of the correct 55^o. As a result the rotation estimates were considerably smaller that the true values. This also resulted in errors in the translation and depth estimates.

It is important to note that while the magnitude of the angle estimates in [74] were too small, the estimated axis of rotation was correct. In fact the results show a linear relationship between true angle and estimated angle. That the rotation values were always smaller than the true values is consistent with the simulation results shown in Figure 3-5. This leads us to the following simple modification to the iterative scheme (Section 3.7.2):

1. For the estimation of motion use a focal length parameter $f << f_{true}$.

2. During the image warping stage use the correct focal length to rotate Images 2 and Image 3 towards Image 1.

This modification required no changes to the structure of the algorithm since the coarse-to-fine and iterative mechanisms were already in place. The results shown in Section 4.3.1 show that the modified algorithm gives accurate rotation estimates even for medium fields of view (55^o).

How does this affect the translation estimates ?

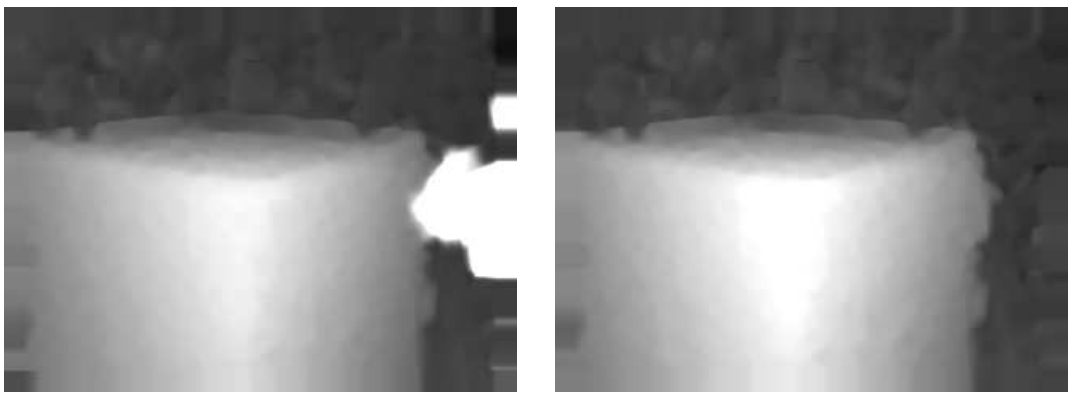
How does using the wrong focal length affect the translation estimates ? Looking at equation (3.20) (and also (3.16), (3.19) and (3.37)) we see that the translations t' and t'' always appear in a dot product with s where:

$$s = \begin{pmatrix} I_x \\ I_y \\ -xI_x - yI_y \end{pmatrix}.$$

The first and second terms of s are not affected by the focal length estimate. Therefore there is no effect on the translation estimate in the X and Y direction, t_x and t_y. In the third term, x and y are supposed to be the normalized image coordinates. If we use the wrong focal length then the estimated Z translation (t_z) will be scaled accordingly.

$$\frac{t_Z}{t_{Ztrue}} = \frac{f}{f_{true}}.$$

Using this simple relationship we can recover the true translation direction from the estimated one even if we use a focal length value which is much smaller than the true value.



(a) (b)

Figure 3-7: *An example of a local depth error (a) fixed by a second pass down the pyramid performing 'depth-only' computation (b). (see text)*

3.7.6 Final Touches - Recompute the Depth Keeping the Motion Constant

Initially, during the first iterations and at the coarsest levels of the pyramid, the motion estimates are not accurate and cause errors in the depth estimates. Sometimes, particularly near the borders of the image, these errors are large enough to affect the new derivatives, I'_t and I''_t in such a way that at the next iteration even when the motion estimates are better the depth values are still badly in error. This tends to be a local problem and does not affect the global motion estimates but it does create local 'holes' in the depth map.

To fix this problem, after computing the motion at the finest level, we go back down the pyramid and recompute the depth while keeping the motion estimate constant. This is in effect becomes a three camera stereo computation. Figure 3-7 shows an example of the depth estimates before and after the second pass.

3.8 Summary of Theoretical Results and Future Work

We have presented a general relationship between the spatio-temporal derivatives of three frames and the ego-motion parameters of the two motions. This relationship was derived first for the general case where the ego-motion model comprises of the 27 coefficients of the trilinear tensor, and then for the small-motion model of Longuet-Higgins & Prazdny.

On practical grounds, these relationships are model-based brightness constraints which provide a linear constraint per pixel in the image — thereby providing a direct structure and motion estimation that cuts through the aperture problem and without prior detection of feature points. The linear constraints of the 27-parameter projective model are degenerate and cannot lead directly to a unique solution for the tensor

coefficients although a unique solution can be found using quadratic admissibility constraints. This added complexity lead us to implement in practice the simpler small-motion model of Longuet-Higgins & Prazdny.

The implementation details of these model-based brightness constraints are important and include four critical elements: (i) embedding of the computations within a coarse-to-fine (Gaussian pyramid) framework, (ii) Newton iterations over the brightness constraint equation, (iii) post-processing of smoothness and surface interpolation for obtaining visually pleasing results, and (iv) obtaining both stability and accuracy by assuming a very short focal length for the motion estimation, but using the correct focal length for the image warping.

In the following chapter we test the algorithm on a set of challenging real image situations, some of which contain very few "good features" (local regions with significant variability of gradient direction) that are necessary for optical flow and discrete point matching algorithms. Yet, we obtained a faithful dense depth map of the scene due to the fact that the algorithm is not hindered by the presence of aperture effects.

3.8.1 Future Work - Collinear Motion

The method, as described so far, fails when the two motions are in the same (or opposite) directions. This is a drawback in many applications (e.g. 3D reconstruction from a monocular image sequence). We present here some early research on ways to overcome this problem. The initial results look promising (Section 4.2.2). We will investigate the pure translation case because if it fails for pure translation, it will also fail if there is some rotation.

For pure translation:

$$I''_t s^\top t' - I'_t s^\top t'' = 0 \tag{3.48}$$

and

$$k s^\top t'' + I''_t = 0 \tag{3.49}$$

$$k s^\top t' + I'_t = 0 \tag{3.50}$$

If the second translation vector t'' is proportional to the translation vector t' then equation (3.49) is simply a scaled version of equation (3.50) adding no new information and the solution is ill conditioned. Another way of interpreting equations (3.50) and (3.49) is that a motion t' in one direction will create a change in the image I'_t which exactly the opposite of the change I''_t induced by a motion t'' of equal magnitude and opposite direction to t'.

The LH&P model which assumes ($\frac{t_z}{Z} << 1$) and was used to derive equations (3.50) and (3.49) is of course not exact unless the motion in the Z direction is zero, but in any case the contribution to the optical flow due to translation in the Z direction is small unless the field of view is very wide. Therefore, for small motion, the LH&P model will be quite accuarte and the equations will be ill-conditioned.

This situation is closely related to the case of reconstruction from line correspondences. There is a critical combination of a line configuration and camera motions which can be stated in the following theorem:

Theorem 2 *Critical configuration of lines and motion*

Let $\mathcal{S}$ be a set of lines in 3D which have a common intersecting line L (i.e., $S \wedge L = 0$ for all $S \in \mathcal{S}$). Let s,s′,s″ be the projections of the line $S \in \mathcal{S}$ in the three views: Image 1, Image 2 and Image 3. Let o, o′ and o″ be the corresponding camera centers. If o, o′ and o″ all lie on a line $S_i \in \mathcal{S}$ the corresponding lines s,s′,s″ do not provide a unique solution to the camera motion.

We currently have no proof, but simulation experiments confirm the theorem. This line configuration is the Linear Line Complex (LLC) which is examined in Chapter 6. In Chapter 6 (Theorem 4) we show that for general motion, the tensor has a linear degeneracy in the case of an LLC, but a unique solution is obtained by taking into account nonlinear constraints. Here, for the specific case where the camera motion is along a line belonging to the LLC, there is a whole family of solutions. The epipole could lie anywhere on the projection of the line L onto the image. Thus, there is one degree of uncertainty in addition to the scale factor ambiguity.

If we take the common to be the line at infinity on the XY plane we get the particular case where all the lines lie on planes which are parallel to the image plane, and the translations are collinear and also parallel to the image plane. This is a critical condition according to Theorem 2 and the direction of translation in the XY plane cannot be recovered. In such a configuration, lines in Image 1 will be parallel to corresponding lines in Image 2 and Image 3 (assuming no rotation).

How does this relate to our case? By using the first order approximation of the optical flow constraint equation to get the point-line-line correspondences, we have created, as an artifact, a situation where all corresponding lines are parallel. In reality this is not the case, unless, all the lines came from planes parallel to the image plane.

A Possible Solution

The Optical Flow Constraint Equation (Section 2.5) was based on the first order Taylor expansion of $I(x, y, t) = I(x', y', t + \delta t)$ where t is used to denote time (not translation). Keeping the second order terms of the Taylor expansion results in:

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \delta x I_x + \delta y I_y + \delta t I_t + \frac{1}{2!}\left(\delta x^2 E_{xx} + \delta y^2 E_{yy} + \delta t^2 E_{tt} + 2\delta x \delta y E_{xy} + 2\delta x \delta t E_{xt} + 2\delta y \delta t E_{yt}\right) + e \quad (3.51)$$

Replacing $\delta x = u'\delta t$ and $\delta y = v'\delta t$ yields:

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t) + u'\delta t I_x + v'\delta t I_y + \delta t I_t + \frac{1}{2!}\left((u\delta t)^2 E_{xx} + (v'\delta t)^2 E_{yy} + \delta t^2 E_{tt} + 2u'v'\delta t^2 E_{xy} + 2u'\delta t^2 E_{xt} + 2v'\delta t^2 E_{yt}\right) \quad (3.52)$$

We notice that two of the second order terms are also linear with u and v. If we keep those two terms, (at this point, apart from convenience, we have no formal justification why we should keep those two terms and not all the 2nd order terms) and apply the constant brightness assumption:

$$I(x, y, t) = I(x + \delta x, y + \delta y, t + \delta t)$$

we get for the first motion:

$$u'(I_x + I_{xt}\delta t) + v'(I_y + I_{yt}\delta t) + I'_t = 0 \tag{3.53}$$

Since $I'_x = I_x + I_{xt}\delta t$ and $I'_y = I_y + I_{yt}\delta t$ this becomes:

$$u'I'_x + v'I'_y + I'_t = 0 \tag{3.54}$$

(The original work by Horn & Schunk used: $u'\frac{(I'_x+I_x)}{2} + v'\frac{(I'_y+I_y)}{2} + I'_t = 0$). Note that I'_x and I'_y are computed at coordinates (x, y) not (x', y'). Similarly for the second motion:

$$u''I''_x + v''I''_y + I''_t = 0 \tag{3.55}$$

Likewise I''_x, I''_y are the brightness gradients in Image 3 at coordinates (x, y). That leads us to the equation:

$$I''_t s'^{\top} t' - I'_t s''^{\top} t'' = 0 \tag{3.56}$$

where:

$$s' = \begin{pmatrix} I'_x \\ I'_y \\ -xI'_x - yI'_y \end{pmatrix} \qquad d'' = \begin{pmatrix} I''_x \\ I''_y \\ -xI''_x - yI''_y \end{pmatrix} \tag{3.57}$$

Since now $s' \neq s''$ the equations are better conditioned. This depends on how much change there was in the images gradients. As we have noted, this depends on whether the surface is parallel to the image plane or not.

A modification is required to the coarse-to-fine framework and the iterative refinement which are described in Section 3.7.4 and Section3.7.2 respectively. For motion estimation, Image 1 is warped (using forward warping) towards Image 2 and Image 3. For depth estimation Image 2 and Image 3 are warped towards Image 1.

We have shown an initial solution to the case of collinear motion with pure translation and have successfully implemented these ideas. It is not clear how to transfer this to the general motion case and this remains an open area of study.