Relation Between the APEX Algorithm and GHA

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Kung and Diamantaras's APEX (Adaptive Principal Components Extraction) algorithm can be summarized with the following equations:

$$
\begin{aligned}
y_j &= w_j^T x && (1)\\
\tilde{y}_j &= y_j + a_j^T Y_{j-1} && (2)\\
Y_{j-1} &= [\tilde{y}_0, \ldots, \tilde{y}_{j-1}]^T && (3)\\
\Delta w_j &= \gamma(\tilde{y}_j x - \tilde{y}_j^2 w_j) && (4)\\
\Delta a_j &= -\gamma(y_j Y_{j-1} + y_j^2 a_j) && (5)
\end{aligned}
$$

where x is the input data, a_j are lateral interaction terms affecting the outputs, the usual outputs are given by y_j, w_j are the feedforward weights to y_j, the outputs including lateral interactions are $\tilde{y}_j$, the vector of the first $j-1$ outputs is given by Y_{j-1}, and γ is a rate term.

Sanger's GHA (Generalized Hebbian Algorithm) can be summarized:

$$
\begin{aligned}
\tilde{x} &= x - \sum_{k=0}^{j-1} w_k^T y_k && (6)\\
\Delta w_j &= \gamma(y_j \tilde{x} - y_j^2 w_j) && (7)\\
&= \gamma(w_j^T x \tilde{x} - y_j^2 w_j) && (8)
\end{aligned}
$$

where in this case the outputs do not include lateral interaction terms. The other variables are as in APEX.

First, note that equation 5 for the lateral interactions in APEX is equivalent to the Widrow-Hoff LMS rule for estimating Y_{j-1} from y_j using the linear coefficients $-a_j$:

$$
\begin{aligned}
\hat{Y}_{j-1} &= -a_j y_j && (9)\\
-\Delta a_j &= \gamma(Y_{j-1} - \hat{Y}_{j-1}) y_j && (10)
\end{aligned}
$$

so at convergence $-a_j y_j$ will be the best linear estimate of Y_{j-1} from y_j.

Now define

$$W_{j-1} = [w_0, ..., w_{j-1}]$$

and since Kung and Diamantaras show that $a_k \rightarrow 0$ for $k < j$, we can write

$$Y_{j-1} \rightarrow W_{j-1}^T x. \qquad (11)$$

This implies that the best estimate of Y_{j-1} from $y_j = w_j^T x$ is given by

$$\hat{Y}_{j-1} = W_{j-1}^T w_j \|w_j\|^{-1} y_j$$

so that equation 5 converges to

$$a_j = -W_{j-1}^T w_j \|w_j\|^{-1} \tag{12}$$

and note that APEX causes $\|w_j\| \to 1$.

Next, substitute back into equation 2 to obtain:

$$\begin{aligned} \tilde{y}_j &= w_j^T x - w_j^T W_{j-1} Y_{j-1} && (13) \\ &= w_j^T (x - \sum_{k=0}^{j-1} w_k^T y_k) && (14) \\ &= w_j^T \tilde{x} && (15) \end{aligned}$$

and substitute this into equation 4 to obtain:

$$\Delta w_j = \gamma (w_j^T \tilde{x} x - \tilde{y}_j^2 w) \tag{16}$$

which is almost exactly equivalent to equation 8 except that the normalization term uses $\tilde{y}_j$ instead of y_j, and the order of terms is exchanged in the first multiplication. But we know that a_j converges to 0, so $\tilde{y}_j$ approaches y_j, and the stable points of GHA and APEX must be the same.