

Symmetry in the Coplanarity Condition

We can rewrite the triple product without difficulty using

$$t = \mathring{\mathbf{r}}\mathring{\mathbf{d}} \cdot \mathring{\mathbf{q}}\mathring{\mathbf{\ell}} = \mathring{\mathbf{r}} \cdot \mathring{\mathbf{q}}\mathring{\mathbf{\ell}}\mathring{\mathbf{d}}^* = \mathring{\mathbf{q}}^*\mathring{\mathbf{r}} \cdot \mathring{\mathbf{\ell}}\mathring{\mathbf{d}}^*. \quad (1)$$

Noting that $\mathring{\mathbf{\ell}}^* = -\mathring{\mathbf{\ell}}$ and $\mathring{\mathbf{r}}^* = -\mathring{\mathbf{r}}$, since $\mathring{\mathbf{r}}$ and $\mathring{\mathbf{\ell}}$ are quaternions with zero scalar parts, we first obtain

$$\boxed{t = \mathring{\mathbf{r}}\mathring{\mathbf{q}} \cdot \mathring{\mathbf{d}}\mathring{\mathbf{\ell}}} \quad (2)$$

We then find by expanding the dot-product for t in terms of the scalar and vector components of $\mathring{\mathbf{q}} = (q, \mathbf{q})$ and $\mathring{\mathbf{d}} = (d, \mathbf{d})$:

$$(\mathbf{d} \cdot \mathbf{r})(\mathbf{q} \cdot \mathbf{\ell}) + (\mathbf{q} \cdot \mathbf{r})(\mathbf{d} \cdot \mathbf{\ell}) + (dq - \mathbf{d} \cdot \mathbf{q})(\mathbf{\ell} \cdot \mathbf{r}) + d[\mathbf{r} \times \mathbf{q} \cdot \mathbf{\ell}] + q[\mathbf{r} \times \mathbf{d} \cdot \mathbf{\ell}]. \quad (3)$$

While

$$\mathring{\mathbf{s}} = \sum_{i=1}^n w_i e_i (\mathring{\mathbf{r}}_i \mathring{\mathbf{d}} \mathring{\mathbf{\ell}}_i^*) \quad \text{and} \quad \mathring{\mathbf{t}} = \sum_{i=1}^n w_i e_i (\mathring{\mathbf{r}}_i^* \mathring{\mathbf{q}} \mathring{\mathbf{\ell}}_i). \quad (4)$$

We also still have the three equations

$$\mathring{\mathbf{q}} \cdot \delta \mathring{\mathbf{q}} = 0, \quad \mathring{\mathbf{d}} \cdot \delta \mathring{\mathbf{d}} = 0, \quad \text{and} \quad \mathring{\mathbf{q}} \cdot \delta \mathring{\mathbf{d}} + \mathring{\mathbf{d}} \cdot \delta \mathring{\mathbf{q}} = 0, \quad (5)$$

all of which we can shuffled around into matrix form

$$\begin{pmatrix} A & B & \mathring{\mathbf{q}} & 0 & \mathring{\mathbf{d}} \\ B^T & C & 0 & \mathring{\mathbf{d}} & \mathring{\mathbf{q}} \\ \mathring{\mathbf{q}}^T & 0^T & 0 & 0 & 0 \\ 0^T & \mathring{\mathbf{d}}^T & 0 & 0 & 0 \\ \mathring{\mathbf{d}}^T & \mathring{\mathbf{q}}^T & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \mathring{\mathbf{d}} \\ \delta \mathring{\mathbf{q}} \\ \lambda \\ \mu \\ \nu \end{pmatrix} = - \begin{pmatrix} \mathring{\mathbf{s}} \\ \mathring{\mathbf{t}} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (6)$$

Note that the upper left 8×8 sub-matrix is the *weighted* sum of flattened dyadic products (as first shown by Žarić, Bärüć, and Lolajž)

$$\sum_{i=1}^n w_i \vec{c}_i \vec{c}_i^T, \quad (7)$$

where the eight component vector $\vec{c}_i$ is given by

$$\vec{c}_i = \begin{pmatrix} \mathring{\mathbf{r}}_i \mathring{\mathbf{d}} \mathring{\mathbf{\ell}}_i^* \\ \mathring{\mathbf{r}}_i^* \mathring{\mathbf{q}} \mathring{\mathbf{\ell}}_i \end{pmatrix} = - \begin{pmatrix} \mathring{\mathbf{r}}_i \mathring{\mathbf{q}} \mathring{\mathbf{\ell}}_i \\ \mathring{\mathbf{r}}_i \mathring{\mathbf{d}} \mathring{\mathbf{\ell}}_i \end{pmatrix}. \quad (8)$$

We conclude that the number of solutions is equal to the number of ways of partitioning the set of variables, namely

$$\binom{n+m-2}{n-1} = \binom{n+m-2}{m-1} = \frac{(n+m-2)!}{(n-1)!(m-1)!} \quad (9)$$

To implement the numerical solution, take a small step $\delta\lambda$ in λ and solve for the increment $\delta\mathbf{x}$ in

$$\frac{d\mathbf{h}}{d\lambda} \delta\lambda + \frac{d\mathbf{h}}{d\mathbf{x}} \delta\mathbf{x} = 0, \quad (10)$$

where $J = (d\mathbf{h}/d\mathbf{x})$ is the Jacobian of $\mathbf{h}$ with respect to $\mathbf{x}$.