The Geometry of Multiple Images

O.D. Faugeras and Q.-T. Luong
with contributions from T. Papadopoulo

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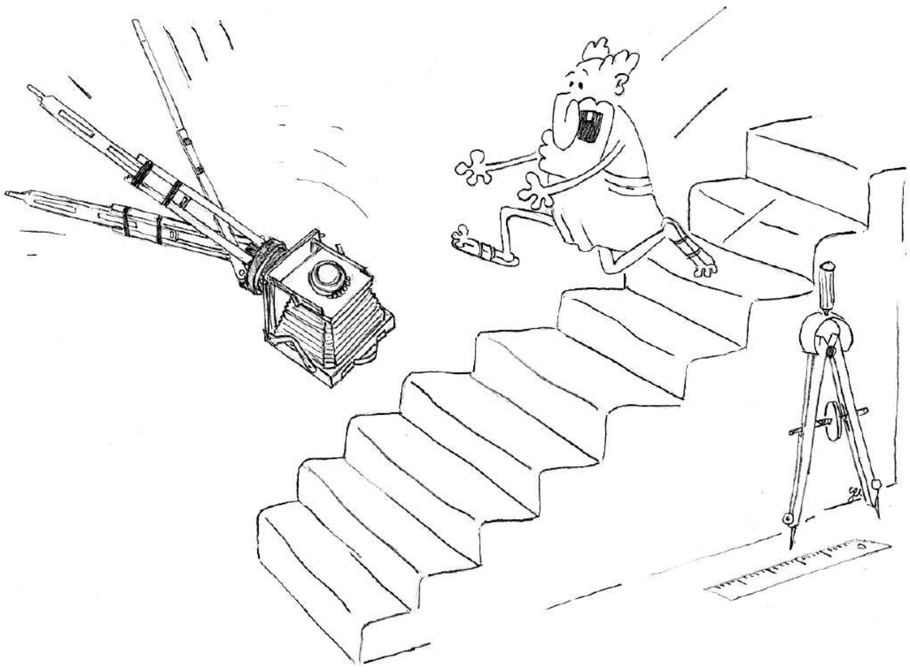
Chapter 12

Self-calibration of a moving camera: from affine or projective calibration to full Euclidean calibration



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12.1 Introduction

We have described the three-dimensional geometry of the world as a stratification of a projective, affine, and Euclidean layers. So far, we have seen in chapters 8 and 8 that with some additional information about the world or the motion of the cameras, we could upgrade the representation from one layer to the next one. In this chapter, we address the problem of upgrading the representation using constraints on the camera's intrinsic parameters. This problem can be alternatively formulated as the recovery of the internal parameters of a camera without the need of a calibration pattern, i.e. an object with known geometry in some three-dimensional Euclidean coordinate system.

We focus in the two first sections on the idea of relating directly the infinity morphism and the Fundamental matrix to intrinsic parameters by elimination of the rigid motion. First, even though we have seen in chapter 6 that the infinity morphism is an affine entity, we show here that it can also provide some Euclidean information, namely the intrinsic parameters of the camera. Second, even though we have also seen in the same chapter that the Fundamental matrix between two views is a projective entity, we show here that it can provide some Euclidean information, i.e. also the intrinsic parameters of the camera.

The situations where we can use the infinity morphism and those where we can use the Fundamental matrix are somewhat different. The infinity morphism can be most easily estimated from point correspondences when the camera's motion is restricted to one keeping the optical center fixed, i.e. the camera is purely rotating around an axis going through that optical center, whereas the Fundamental matrix is important as soon as there is enough translation in the motion of the camera and the distance between the positions of the optical centers is significant with respect to the distances of the 3D points in the scene.

The basic idea is that, once we have estimated the infinity morphism between two views if the optical centers are identical, or the Fundamental matrix if they are different, we can use their relations, described in chapter 6, with the internal parameters of the camera to actually constrain those and even, sometimes, to compute them. The computation of the internal parameters from the infinity morphism can be considered an upgrade from affine to Euclidean calibration while the computation of those parameters from the Fundamental matrix is an upgrade from projective to Euclidean calibration. A major role is played in this chapter by the Kruppa matrix introduced in chapter 5. It is in fact this matrix that it is convenient to recover in both cases rather than the intrinsic parameters directly.

Section 12.2 introduces the relation between the Kruppa matrices of two views and the infinity morphism and shows that this relation provides in general four independent equations between the coefficients of the unknown Kruppa matrices. When those are identical, i.e. when the intrinsic parameters of the camera have not changed between the views, this is sufficient to recover four intrinsic parameters but insufficient to recover five. In the case where the skew is known, e.g. equal to 0, we show that two views provide a unique solution for the Kruppa matrix and hence for the calibration of the camera. In the case where the skew is not known, we show that, in general, three views are sufficient to uniquely recover the Kruppa matrix. To illustrate the theory, we give a real example and also show the application of the method to the construction of a mosaic.

Section 12.3 introduces various relations between the Fundamental matrix and the internal parameters of a pair of cameras. It starts with the idea, historically important, to relate the Fundamental and the Essential matrices. We show that, although conceptually simple, that relation yields one complicated algebraic equation between the intrinsic parameters of the two cameras. We then show that by taking the full-fledged projective viewpoint advocated in this book and introducing the absolute conic, we come up with a set of equations between the Kruppa matrices of the two views which are algebraically simpler than the one obtained with the previous approach. We have named these equations after Erwin Kruppa because of his pioneering contributions in the use of projective geometry to analyse perspective projection. There are three such equations of which only two are algebraically independent and we show that in general three views provide six independent equations that are sufficient to recover the five intrinsic parameters in the case where they are identical in the three views.

Section 12.4 investigates practical methods for self-calibration based on the Kruppa equations. These methods solve exclusively for the intrinsic parameters. Once there parameters are obtained, the motion and the structure could be determined. It is shown that the methods for estimating the uncertainty of these Fundamental matrices described in chapter 7 play an important role in the accurate computation of the intrinsic parameters.

Section 12.5 is devoted to methods who solve not only for the intrinsic parameters (which are Euclidean entities), but also for the equation of the plane at infinity (an affine entity). We we first pursue our investigation of the constant intrinsic parameter case for the simpler case of affine cameras, before turning out to the case of perspective projection for which we describe several methods. The section ends with a method that does not assume that the intrinsic parameters are constant in all views but only that some of them are somewhat constrained, e.g. that the cameras are zero skew or that their aspect ratio is known. The method is based upon the estimation of the quadric dual to the absolute conic.

While motion was always eliminated in the approaches previously described, resulting in a relatively small number of unknowns, the methods described in section 12.6 all involve simultaneous determination of the intrinsic parameters and motion parameters. We first use the epipolar constraints, before describing an approach based directly on the projection equations, which in addition computes simultaneously the structure. The rest of the section is devoted to the description of the application of the previous techniques to the construction of 3D Euclidean models of a scene from photographs, and to match moving, i.e the insertion of synthetic objects in sequences of images of real scenes.

Section 12.7 ends the chapter with a discussion of the problem of degeneracy in self-calibration. Theoretically it is important to characterize the situations in which the equations that have been presented in the previous two sections become degenerate in the sense that they do not yield a unique solution for the Kruppa matrix or the intrinsic parameters, because it is part of our understanding of these equations. Practically it is also clearly important to know what are the camera motions that are to be avoided if we want our algorithms to estimate quickly and reliably the calibration parameters. We show that the problem of degeneracy is intrinsic (no pun intended) in the sense that it is algorithm independent: the degeneracy comes from the fact that we use a finite number of views to recover the equation of the image of the absolute conic. But we also show that, beside these intrinsic degeneracies, each particular method, e.g. the one based on the Kruppa equations, adds in general more degeneracies.

12.2 From affine to Euclidean

We start with the special case where the infinity collineation $\mathbf{H}_\infty$ is known. This case is important because, as seen in chapter 6, it corresponds to the situation where the camera is rotating with respect to an axis going through its optical center or to the situation where we have been able to recover the affine structure of the environment.

12.2.1 Theoretical analysis

We first show that the knowledge of $\mathbf{H}_\infty$ places some constraints on the internal parameters of the two views through the Kruppa matrix defined in section 5.3.2:

Proposition 12.1 *Given the infinity collineation $\mathbf{H}_\infty$ between two views, the relation between the Kruppa matrices is the following:*

$$\mathbf{K}' \simeq \mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T \tag{12.1}$$

Proof : We know from proposition 6.25 that $\mathbf{H}_\infty \simeq \mathbf{A}'\mathbf{R}\mathbf{A}^{-1}$, therefore, since $\mathbf{R}\mathbf{R}^T = \mathbf{I}_3$, $\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T \simeq \mathbf{K}'$, where $\mathbf{K}$ and $\mathbf{K}'$ are the two Kruppa matrices. Another, more geometric, way of seeing this is to notice that since Ω is by definition in the plane at infinity Π_∞, then ω and ω' or rather ω^* and ω'^* are image of one another through the infinity collineation H_∞: for all line l' belonging to ω'^*, i.e. such that $\mathbf{l}'^T\mathbf{K}'\mathbf{l}' = 0$, its image through the infinity collineation is $\mathbf{H}_\infty^T \mathbf{l}'$ and belongs to ω^*: $\mathbf{l}'^T \mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T \mathbf{l}' = 0$, which implies equation (12.1). □

This equation implies the equality, up to scale, of the two symmetric matrices $\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T$ and $\mathbf{K}'$. In the case where the internal parameters are the same in the two views, i.e when $\mathbf{K} = \mathbf{K}'$, this yields apparently five equations between those internal parameters. A natural question to ask is whether these constraints are sufficient to recover the intrinsic parameters of the camera. The answer to this question is unfortunately no, as shown in proposition 12.2. We start with a lemma.

Lemma 12.1 *Let $\mathbf{X}$ be an unknown symmetric matrix and $\mathbf{R}$ be a rotation matrix, $\mathbf{R} \neq \mathbf{I}_3$. Let $\mathbf{r}$ be a unit vector parallel to the axis of rotation of $\mathbf{R}$. The solutions of the matrix equation*

$$\mathbf{R}\mathbf{X}\mathbf{R}^T - \mathbf{X} = \mathbf{0} \tag{12.2}$$

fall in two categories:

1. *if $\mathbf{R}$ is a rotation of 180 degrees:*

$$\mathbf{X} = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \mu \mathbf{r}\mathbf{r}^T \tag{12.3}$$

   where $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors orthogonal to $\mathbf{r}$, λ_1, λ_2 and μ are arbitrary real numbers.

2. *if* $\mathbf{R}$ *is not a rotation of 180 degrees: the solutions form a two-dimensional vector subspace* $\mathcal{V}_{\mathbf{r}} = span(\mathbf{I}_3, \mathbf{r}\mathbf{r}^T)$ *of the six-dimensional vector space of the symmetric matrices, i.e:*

$$\mathbf{X} = \lambda \mathbf{I}_3 + \mu \mathbf{r}\mathbf{r}^T \tag{12.4}$$

where λ *and* μ *are arbitrary real numbers.*

Proof : Let us rewrite equation (12.2)

$$\mathbf{RX} = \mathbf{XR}$$

This shows that $\mathbf{r}$ is an eigenvector of $\mathbf{X}$ since $\mathbf{RXr} = \mathbf{XRr} = \mathbf{Xr}$ and that $\mathbf{R}$ must leave the other eigenspaces of $\mathbf{X}$ invariant since for each eigenvector $\mathbf{u}$ of $\mathbf{X}$, $\mathbf{RXu} = \lambda \mathbf{Ru} = \mathbf{XRu}$.

Consider the eigenspaces orthogonal to $\mathbf{r}$. Two cases can occur. The first case is when there are two orthogonal eigenspaces corresponding to the two distinct eigenvalues λ_1 and λ_2 and generated by the two unit norm orthogonal vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Since $\mathbf{Ru}_1$ is an eigenvector of $\mathbf{X}$ and is in the plane $span(\mathbf{u}_1, \mathbf{u}_2)$ (because $\mathbf{R}$'s axis is orthogonal to that plane), the only possibility is that $\mathbf{Ru}_1 = \pm \mathbf{u}_1$. In the first case, $\mathbf{R}$ is the identity, in the second it is a rotation of 180 degrees and $\mathbf{X}$ takes the form (12.3).

The second case to be considered is when there is an eigenplane orthogonal to $\mathbf{r}$ corresponding to the eigenvalue λ. The restriction of $\mathbf{X}$ to this eigenplane is $\lambda \mathbf{I}_3$ and $\mathbf{X}$ takes the form (12.4). □

We are now ready to state the

Proposition 12.2 *The solutions in* $\mathbf{K}$, *a symmetric matrix, of the matrix equation:*

$$\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T = \mathbf{K} \tag{12.5}$$

where $\mathbf{H}_\infty$ *is of rank 3, has all eigenvalues of unit magnitude, two of them complex conjuguate, are given by:*

$$\mathbf{K} = \lambda \mathbf{A}\mathbf{A}^T + \mu (\mathbf{Ar})(\mathbf{Ar})^T \tag{12.6}$$

where λ *and* μ *are arbitrary constants and* $\mathbf{r}$ *is a unit vector satisfying* $\mathbf{Rr} = \mathbf{r}$.

Proof : From proposition 6.26 we can assume that $\mathbf{H}_\infty = \mathbf{ARA}^{-1}$ and transform the equation (12.5) into:

$$\mathbf{R}(\mathbf{A}^{-1}\mathbf{K}\mathbf{A}^{-T})\mathbf{R}^T - \mathbf{A}^{-1}\mathbf{K}\mathbf{A}^{-T} = \mathbf{0}$$

which expresses the fact that $\mathbf{K}$ is a solution of the original equation if and only if $\mathbf{A}^{-1}\mathbf{K}\mathbf{A}^{-T}$ is a solution of:

$$\mathbf{R}\mathbf{X}\mathbf{R}^T - \mathbf{X} = 0$$

We then just have to apply lemma 12.1 to complete the proof. □

Note that we have eliminated, as non generic, the case where the rotation between the views is 180 degrees.

An interpretation of this proposition is that the knowledge of the infinity morphism between two views taken with the same camera imposes in general four constraints on the five intrinsic parameters. A special case of interest is the case where the number of intrinsic parameters is reduced to four, e.g. when the skew is zero, a case of practical interest. We state the result in the following proposition:

Proposition 12.3 *When the camera is zero-skew, the knowledge of* $\mathbf{H}_\infty$ *yields in general a unique solution for the matrix* $\mathbf{K}$.

Proof : We know from proposition 12.2 that the solutions of the system of linear equations $\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T = \mathbf{K}$ form a vector space of dimension 2. Let $\mathbf{K}_1$, $\mathbf{K}_2$ be a basis of that vector subspace (in practice, it can be found for example by SVD, see section 12.2.2). Hence, all solutions are of the form

$$\mathbf{K} = \lambda_1 \mathbf{K}_1 + \lambda_2 \mathbf{K}_2 \tag{12.7}$$

Note that this basis is in general different from the basis $(\mathbf{A}\mathbf{A}^T, \mathbf{Ar}(\mathbf{Ar})^T)$ but when λ_1 and λ_2 vary, $\mathbf{K}$ takes all the values given by equation (12.6). This family of symmetric matrices contains one singular matrix, $\mathbf{Ar}(\mathbf{Ar})^T$, obtained for $\lambda = 0$ and, since $\mathbf{A}\mathbf{A}^T$ is nonsingular, it is the only singular matrix in the family. Let us now use the fact that the camera is zero skew. According to the analysis done in section 5.3.2 and to equation (5.37), $\mathbf{K}$ must satisfy the constraint

$$K_{13}K_{23} - K_{33}K_{12} = 0 \tag{12.8}$$

It can be verified that the matrix $\mathbf{Ar}(\mathbf{Ar})^T$ satisfies this equation.

The lefthand side of (12.8) is a second degree homogeneous polynomial in λ_1 and λ_2 which has in general two roots in $\frac{\lambda_1}{\lambda_2}$, yielding two solutions for $\mathbf{K}$. Among the two solutions, one of them has to be proportional to $\mathbf{Ar}(\mathbf{Ar})^T$. The second solution must therefore be proportional to $\mathbf{A}\mathbf{A}^T$. The scale factor can be accounted for by computing its Cholewski decomposition. □

Let us now return to the general case of five intrinsic parameters. We can hope that if we perform two motions of the camera such that the corresponding rotation matrices are sufficiently different, in a sense to be made more precise, we should be able to recover the intrinsic parameters. We start with the following proposition:

Proposition 12.4 *Let $\mathbf{R}_i, i = 1,2$ be two rotation matrices such that $\mathbf{R}_1 \neq \mathbf{I}_3$, $\mathbf{R}_2 \neq \mathbf{I}_3$ and furthermore none of them is a rotation of 180 degrees. Let $\mathbf{X}$ be an unknown symmetric matrix. The solutions in $\mathbf{X}$ of the system of two equations:*

$$\mathbf{R}_i \mathbf{X} \mathbf{R}_i^T - \mathbf{X} = \mathbf{0} \quad i = 1,2$$

are given by:

$$\mathbf{X} \in \begin{cases} span(\mathbf{I}_3) \text{ if the two rotations have different axes} \\ span(\mathbf{I}_3,\, \mathbf{r}_1\mathbf{r}_1^T) = span(\mathbf{I}_3,\, \mathbf{r}_2\mathbf{r}_2^T) \text{ otherwise} \end{cases}$$

Proof : The solutions of the system of two linear matrix equations lie in the intersection of the two vector spaces $\mathcal{V}_{\mathbf{r}_i}$ $i = 1,2$ of the solutions of each equation. It is easy to see that if the two rotations have different axes this intersection is reduced to the one-dimensional vector space generated by $\mathbf{I}_3$ whereas in the other case the two vector spaces are identical. □

We are now in a position to prove our main result:

Theorem 12.1 *Let $\mathbf{H}_{\infty i}, i = 1,2$ be two infinity collineations, i.e. such that there exist two rotation matrices (none of them of 180 degrees) $\mathbf{R}_i, i = 1,2$ such that $\mathbf{H}_{\infty i}, i = 1,2$ be congruent to $\mathbf{R}_i, i = 1,2$ with the* same *matrix $\mathbf{A}$. Let $\mathbf{K}$ be an unknown symmetric matrix. The solutions $\mathbf{K}$ of the system of two equations:*

$$\mathbf{H}_{\infty i} \mathbf{K} \mathbf{H}_{\infty i}^T - \mathbf{K} = \mathbf{0} \quad i = 1,2$$

are given by:

$$\mathbf{K} \in \begin{cases} span(\mathbf{A}\mathbf{A}^T) \text{ if the two rotations have different axes} \\ span(\mathbf{A}\mathbf{A}^T,\, \mathbf{A}\mathbf{r}_1(\mathbf{A}\mathbf{r}_1^T) = span(\mathbf{A}\mathbf{A}^T,\, \mathbf{A}\mathbf{r}_2(\mathbf{A}\mathbf{r}_2^T)) \text{ otherwise} \end{cases}$$

Proof : The proof follows from proposition 12.4 using the same argument as in proposition 12.2. □

This theorem tells us that unless we happen to rotate around the same axis the five intrinsic parameters are uniquely determined by two views, provided that we can estimate the two infinity morphisms.

12.2.2 Practical computation

Equation (12.5) can be rewritten as a system of six linear equations in the six unknown coefficients of the symmetric matrix $\mathbf{K}$:

$$\boldsymbol{\mathcal{M}}\mathbf{k} = \mathbf{0} \tag{12.9}$$

Because of the noise in $\boldsymbol{\mathcal{M}}$, in practice one solves the following minimization problem:

$$\min_{\mathbf{k}} \|\boldsymbol{\mathcal{M}}\mathbf{k}\|^2 \qquad \text{subject to} \qquad \|\mathbf{k}\| = 1 \tag{12.10}$$

Theoretically, according to proposition 12.2, the rank of matrix $\boldsymbol{\mathcal{M}}$ is four in general. Therefore, one expects two of the singular values of matrix $\boldsymbol{\mathcal{M}}$ to be significantly smaller than the other four (see the next section for an example). The solutions of problem (12.10) are therefore taken as being in the two-dimensional vector space generated by the two eigenvectors of the symmetric 6×6 matrix $\boldsymbol{\mathcal{M}}^T\boldsymbol{\mathcal{M}}$ corresponding to the two smallest eigenvalues.

If one has more than two views, as in proposition 12.4, one applies the same analysis and solves the system of linear equations

$$\boldsymbol{\mathcal{M}}_n\mathbf{k} = \mathbf{0} \tag{12.11}$$

$\boldsymbol{\mathcal{M}}_n$ is an $6n \times 6$ matrix obtained by concatenating the n matrices provided by the knowledge of the infinity collineation between n pairs of views. In practice one solves:

$$\min_{\mathbf{k}} \|\boldsymbol{\mathcal{M}}_n\mathbf{k}\|^2 \qquad \text{subject to} \qquad \|\mathbf{k}\| = 1 \tag{12.12}$$

Theoretically, according to theorem 12.1, the rank of (12.11) is in general five as soon as n is larger than or equal to 2. The solution of (12.12) is therefore taken in the one-dimensional vector subspace generated by the eigenvector of the symmetric 6×6 matrix $\boldsymbol{\mathcal{M}}_n^T\boldsymbol{\mathcal{M}}_n$ corresponding to the smallest eigenvalue.

12.2.3 A numerical example

The knowledge of the infinity homography, enables us to perform (partial) self-calibration from two views in the case of constant intrinsic parameters, but also to deal with the case of variable intrinsic parameters, by propagating affine, and then Euclidean information. However, it has been reported [142, 442] that the methods based on the infinity homography matrix are very sensitive to the precision of determination of the homography, and that in practice, several views are necessary to obtain good results.

The following example may help to ground the previous propositions in reality. Three images of an indoor scene shown Fig. 12.1 are used. The camera is zooming: The focal length has been changed from 8mm to 12mm (read on the lens barrel) between the second and the third view.

Figure 12.1: Three images taken by a camera, with epipolar lines shown. The camera has zoomed between the second and the third view, thereby changing the internal parameters.

Partial self-calibration of a camera from the infinity homography
The infinity homography matrix between the two first views has been determined using a calibration object with a good precision:

$$\mathbf{H}_{\infty 12} = \begin{bmatrix} 1.72134 & -.172135 & -218.805 \\ .758041 & 1.59733 & -620.388 \\ .00109138 & .000592603 & 1.00000 \end{bmatrix}$$

The three eigenvalues of $\mathbf{H}_{\infty 12}$ have the same norm, 1.57575, thus we conclude that the intrinsic parameters remain constant between image 1 and image 2. We divide the entries of $\mathbf{H}_{\infty 12}$ by 1.57575 so that the determinant is now 1.

The singular values σ_i for the matrix $\boldsymbol{\mathcal{M}}$ in Eq. (12.9) are:

$$165495 \qquad 469 \qquad 2.02 \qquad 0.19 \qquad 1.610^{-9} \qquad 910^{-13}$$

We conclude that this matrix is of rank 4, as expected. Taking the last two columns of the orthogonal matrix $\mathbf{V}$ such that the singular value decomposition of $\mathcal{M}$ is $\mathbf{U}\mathrm{diag}(\sigma_1, \ldots, \sigma_6)\mathbf{V}^T$, gives the space of solutions:

$$
\begin{array}{ll}
K_{11} = -.16481 - .75667t & K_{12} = -.38670 + .63966t \\
K_{22} = -.90735 - .13517t & K_{23} = -.00081 + .000855t \\
K_{33} = -.13351 - .80646t & K_{13} = -.2335810^{-5} - .0009565t
\end{array}
$$

The zero-skew constraint of Eq (5.37) leads to the two solutions: $t_1 = .434624$, $t_2 = 3.9219$. After substitution, we find that $\det(\mathbf{K}_1) = -.5610^{-6}$ and $\det(\mathbf{K}_2) = -.4610^{-17}$. Therefore, the solution to be kept is t_1, and by applying the formulas in proposition 5.22, we recover:

$$\alpha_u = 481, \ \alpha_v = 711, \ u_0 = 248, \ v_0 = 260$$

Propagation of affine and Euclidean information Having obtained somewhere in the image sequence Euclidean information, we show know how to propagate it on the whole sequence, thus updating the calibration of a *zooming* camera.

After determining the three Fundamental matrices between the images, we are able to obtain, using proposition 11.4 in chapter 11, the other infinity collineations from $\mathbf{H}_{\infty 12}$. In particular:

$$\mathbf{H}_{\infty 23} = \begin{bmatrix} 1.07992 & .302292 & -35.2662 \\ -.905154 & .959146 & 413.566 \\ -.0000364269 & -.000355526 & 1.00000 \end{bmatrix}$$

The norms of the eigenvalues are $1.21, 1.21, 1.0$, thus we conclude that the intrinsic parameters have changed between image 2 and image 3. The zooming thus took place between these two images. Applying formula (12.1) gives immediately the new matrix $\mathbf{K}'$, from which the following parameters are computed:

$$\alpha_u = 642, \ \alpha_v = 950, \ u_0 = 248, \ v_0 = 263$$

It can be verified that they correspond to the variation of focal length described previously: We have $\frac{\alpha_u^{(3)}}{\alpha_u^{(1)}} \sim \frac{\alpha_v^{(3)}}{\alpha_v^{(1)}} \sim \frac{f^{(3)}}{f^{(1)}} = 1.5$.

We have thus updated the calibration of the intrinsic parameters. As explained in chapter 11, we can go even further and recover a set of Euclidean perspective projection matrices which allow us to recover the structure of the scene up to a similarity transformation.

12.2.4 Application to panoramic mosaicing

In Sec. 6.1.2 we have explaned how to build a planar mosaic using homographies. There was no need for calibration, but on the other hand, we had to reproject all the images onto a planar surface. If we wish to build a mosaic which encompasses a panoramic field of view (larger than 180 degrees), this is no longer reasonable. By recovering the Euclidean parameters (intrinsic parameters and rotations), we can reproject the images onto any surface, including a cylinder or a sphere.

Figure 12.2 show a set of nine views which have been acquired on the site of INRIA in Sophia-Antipolis with a Kodak DCS160 digital camera from an approximately constant viewpoint. They cover 360 degrees of field of view. The intrinsic parameters have been computed using the technique described previously. The relative rotations have also been computed using techniques described in chapter 11. The images have then been mapped to a cylinder which is shown 12.3. Note that the left and right sides of the resulting rectangular image match perfectly, a demonstration of the quality of the calibration.

If besides panning the camera, we take images by tilting it, with a sufficient number of images we can cover the whole sphere. This idea results in the spherical mosaics shown in Fig. 12.4, where we show the two half-hemispheres. It can be noted that this representation makes it possible to synthetize any perspective view taken from the initial viewpoint in any direction, as illustrated in Fig. 12.5.

12.3 From projective to Euclidean

In this section, we study the case of perspective projection. There is an important link between the Fundamental matrix and the intrinsic parameters, which comes from the fact that a general projection matrix can be decomposed in intrinsic parameters and a rigid displacement (see chapter 5). Thus, the knowledge of the Fundamental matrix puts some constraints on the intrinsic parameters. This section studies these constraints in great detail and shows how they can be used for self-calibration.

We thus consider two pinhole cameras which perform perspective projection. We denote by C and C' the two optical centers (assumed to be different) and by $\mathbf{A}$ and $\mathbf{A}'$ the matrices of the internal parameters defined in chapter 5.

Figure 12.2: Nine images of INRIA at Sophia-Antipolis acquired from the same viewpoint.

Figure 12.3: The mosaic built from the twelve images of figure 12.2. Courtesy **Sylvain Bougnoux**, INRIA.

Figure 12.4: The two hemispheres of the spherical mosaic of the INRIA parking lot. Courtesy **Sylvain Bougnoux**, INRIA.

12.3.1 The rigidity constraints: algebraic formulations using the essential matrix

The first natural idea is to use the relation described in proposition 6.27 between the Fundamental matrix and the Essential matrix and the constraints (6.34) satisfied by the Essential matrix to come up with constraints between the internal parameters of two views.

Proposition 12.5 *The knowledge of the Fundamental matrix between the two views yields one algebraic polynomial relation of degree 8 between the coefficients of* $\mathbf{A}$ *and* $\mathbf{A}'$.

Proof : We know from proposition 6.27 that the following relation

$$\mathbf{E} \simeq \mathbf{A}'^T \mathbf{F} \mathbf{A} \tag{12.13}$$

holds between the E-matrix, the F-matrix and the internal parameters. But we know from chapter 6 that an E-matrix satisfies two algebraic constraints, the so-called Huang-Faugeras equations:

$$\det(\mathbf{E}) = 0 \qquad f(\mathbf{E}) = \frac{1}{2}\mathrm{trace}^2(\mathbf{E}\mathbf{E}^T) - \mathrm{trace}(\mathbf{E}\mathbf{E}^T)^2 = 0 \tag{12.14}$$


Figure 12.5: A semi-hemisphere of the mosaic of the Arcades square, and a synthetized perspective view. Courtesy **Sylvain Bougnoux**, INRIA.

As we have $\det(\mathbf{F}) = 0$, the first condition is automatically satisfied, and does not yield any valuable constraint in our framework, thus we are left with only one polynomial constraint, the second condition, which can be seen to be a polynomial of degree 8 on the coefficients of $\mathbf{A}$ and $\mathbf{A}'$. □

A second expression of the rigidity constraints has been presented by Trivedi [383]. If $\mathbf{E}$ is an essential matrix, the symmetric matrix $\mathbf{S} = \mathbf{E}\mathbf{E}^T$, which *a priori* has six independent entries, depends only on the three components of $\mathbf{t}$:

$$\mathbf{S} \equiv \mathbf{E}\mathbf{E}^T = -[\mathbf{t}]_\times{}^2 = \begin{bmatrix} t_2^2 + t_3^2 & -t_1 t_2 & -t_1 t_3 \\ -t_2 t_1 & t_3^2 + t_1^2 & -t_2 t_3 \\ -t_3 t_1 & -t_3 t_2 & t_1^2 + t_2^2 \end{bmatrix} \tag{12.15}$$

The matrix $\mathbf{S}$ has thus a special structure which is described in the following proposition:

Proposition 12.6 (Trivedi) *The elements of the matrix* $\mathbf{S}$ *are related by the three polynomial relations:*

$$4S_{ij} - (trace(\mathbf{S}) - 2S_{ii})(trace(\mathbf{S}) - 2S_{jj}) = 0, \qquad 1 \le i < j \le 3 \tag{12.16}$$

of which only two are independent.

The proof of this proposition can be found in [383] and [228].

It can be seen that the two Trivedi relations provide two polynomial relations of degree 8 between the coefficients of $\mathbf{A}$ and $\mathbf{A}'$ and a natural question is whether those conditions are the same as those obtained from the Huang-Faugeras equations. Fortunately, the answer to this question is positive:

Proposition 12.7 *The Huang-Faugeras and the Trivedi equations are equivalent.*

Proof : Let first suppose that we have (12.15). Then follows immediately $\det(\mathbf{E}\mathbf{E}^T) = 0$, and thus the first condition $\det(\mathbf{E}) = 0$ is satisfied. Adding the three equations (12.16), yields :

$$4(S_{12}^2 + S_{13}^2 + S_{23}^2) + S_{11}^2 + S_{22}^2 + S_{33}^2 - 2(S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11}) = 0$$

Since the matrix $\mathbf{S}$ is symmetric, the first term can be replaced by: $4(S_{12}S_{21} + S_{13}S_{31} + S_{23}S_{32})$, and a simple computation shows that it is identical with the Huang-Faugeras condition:

$$\mathrm{trace}^2(\mathbf{S}) - 2\mathrm{trace}(\mathbf{S}^2) = 0$$

Conversely, let us then assume that the Huang-Faugeras conditions are satisfied. They are equivalent to the fact that the matrix $\mathbf{E}$ has one zero singular value and two equal nonzero singular values σ. By definition, there exists an orthogonal matrix $\mathbf{U}$ such as:

$$\mathbf{S} = \mathbf{E}\mathbf{E}^T = \mathbf{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix} \mathbf{U}^T$$

This matrix equality can be expanded as:

$$\mathbf{S} = \sigma^2 (U_{i2}U_{j2} + U_{i3}U_{j3})_{1 \le i,j \le 3}$$

Since $\mathbf{U}$ is orthogonal, its row vectors are orthonormal:

$$U_{i2}U_{j2} + U_{i3}U_{j3} = \begin{cases} -U_{i1}U_{j1} & \text{if } i \neq j \\ 1 - U_{i1}^2 & \text{if } i = j \end{cases}$$

The diagonal element $1 - U_{11}^2$ (resp. $1 - U_{21}^2$, $1 - U_{31}^2$) can be rewritten, because the column vectors of $\mathbf{U}$ are orthonormal, $U_{31}^2 + U_{21}^2$ (resp. $U_{11}^2 + U_{31}^2$, $U_{21}^2 + U_{11}^2$), which shows that $\mathbf{S}$ has exactely the form (12.15). □

Unfortunately, the two independent Trivedi equations which are equivalent to the second Huang and Faugeras condition are not simpler than this one: They all yield algebraic constraints which are polynomials of degree 8 in the coefficients of $\mathbf{A}$ and $\mathbf{A}'$ (the intrinsic parameters) and thus are not quite suitable for practical computation, or even theoretical study. It is why we are going to consider a *geometrical* interpretation of the rigidity constraint which yields low-order polynomial constraints.

12.3.2 The Kruppa equations: a geometric interpretation of the rigidity constraint

Geometric and algebraic derivation of the Kruppa equations: Let us consider an epipolar plane Π, which is tangent to Ω. Then the corresponding epipolar lines l and l' are tangent to the images ω and ω' of Ω, respectively. It follows that the two tangents to ω drawn from the epipole $\mathbf{e}$ are in correspondence under the epipolar transformation with the two tangents to ω' drawn from the epipole $\mathbf{e}^{'}$, as illustrated by figure 12.6.

Let $\mathbf{K}$ and $\mathbf{K}'$ be the Kruppa matrices, introduced in chapter 6, in the first and second image Eq. 5.34. The fact that the camera is moving rigidly implies two polynomial constraints on the coefficients of $\mathbf{K}$ and $\mathbf{K}'$:

Proposition 12.8 *The two Kruppa matrices satisfy the following relation:*

$$\mathbf{FKF}^T \simeq [\mathbf{e}']_\times \mathbf{K}' [\mathbf{e}']_\times \tag{12.17}$$

which is equivalent to two polynomial equations of degree 2 in the coefficients of $\mathbf{K}$ *and* $\mathbf{K}'$. *These equations are called the* Kruppa *equations.*

Proof : The epipolar line $l' = (e', y')$ is tangent to ω' iff:

$$(\mathbf{e}' \times \mathbf{y}')^T \mathbf{K}' (\mathbf{e}' \times \mathbf{y}') = -\mathbf{y}'^T [\mathbf{e}']_\times \mathbf{K}' [\mathbf{e}']_\times \mathbf{y}' = 0 \tag{12.18}$$

The epipolar line corresponding to the point $\mathbf{y}'$ is $\mathbf{F}^T\mathbf{y}'$ and is tangent to ω iff:

$$\mathbf{y}'^T \mathbf{FKF}^T \mathbf{y}' = 0 \tag{12.19}$$

Writing that (12.18) and (12.19) are equivalent yields the so-called Kruppa equations.

Another, purely algebraic, way of obtaining those equations is to start with equation (12.1):

$$\mathbf{K}' \simeq \mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T,$$

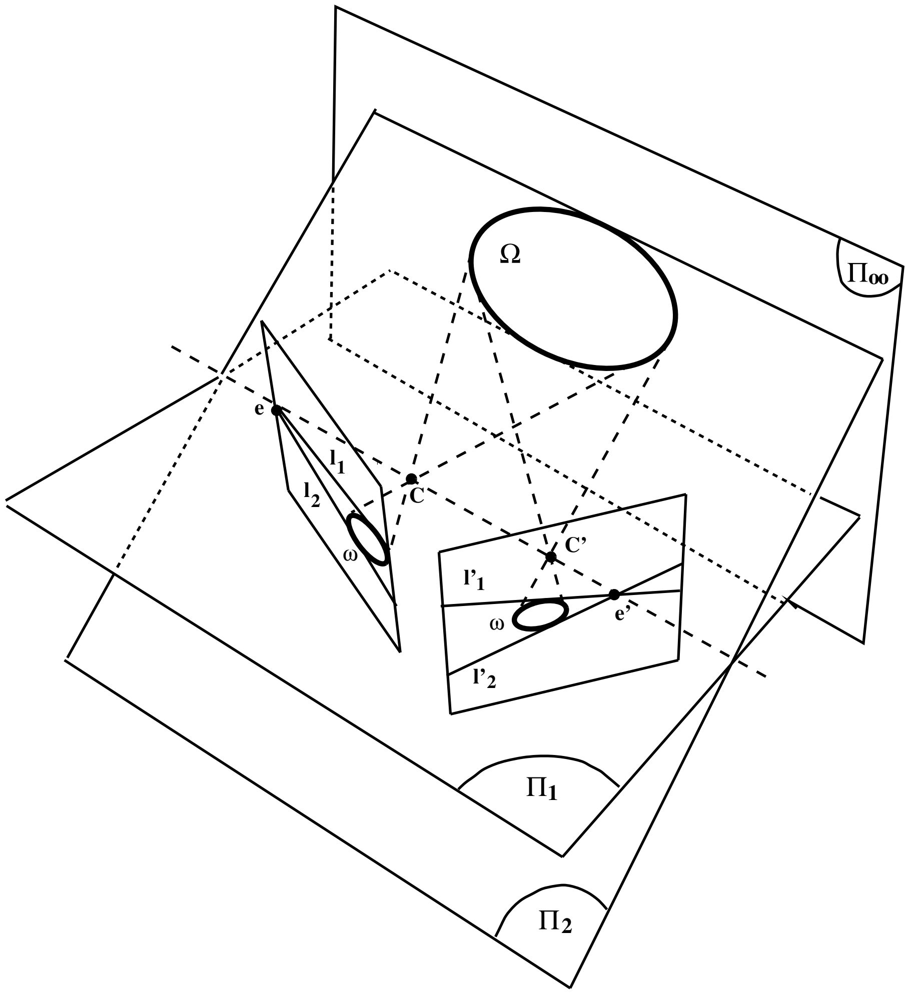


Figure 12.6: The absolute conic and the epipolar transformation.

to multiply left and right by $[\mathbf{e}']_\times$

$$[\mathbf{e}']_\times \mathbf{K}'[\mathbf{e}']_\times \simeq [\mathbf{e}']_\times \mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^T [\mathbf{e}']_\times,$$

and to use equation (6.16):

$$\mathbf{F} \simeq [\mathbf{e}']_\times \mathbf{H}_\infty$$

In order to obtain the two Kruppa equations, we assume for simplicity that the epipole e' is not at infinity, i.e. that its last coordinate is not zero. We can therefore parameterize the epipolar lines (e', y') by their point of intersection with the line at infinity: $\mathbf{y}'_\infty = [1, \tau, 0]^T$. The relations (12.18) (resp. (12.19)) take the form $P(\tau) = k_0 + k_1\tau + k_2\tau^2 = 0$ (resp. $P'(\tau) = k'_0 + k'_1\tau + k'_2\tau^2 = 0$), and the Kruppa equations express the fact that the coefficients of those two polynomials in τ are proportional:

$$k_0 : k_1 : k_2 = k'_0 : k'_1 : k'_2 \tag{12.20}$$

This yields two algebraically independent equations among the three equations

$$Eq_1 = k_1k'_2 - k'_1k_2 = 0 \qquad Eq_2 = k_2k'_0 - k'_2k_0 = 0 \qquad Eq_3 = k_0k'_1 - k'_0k_1 = 0 \tag{12.21}$$

□

The nice thing with the Kruppa equations is that since the coefficients k_i (resp. k'_i) depend linearly on the entries of $\mathbf{K}$ (resp. $\mathbf{K}'$), they are only of degree *two* in these entries, and therefore of degree 4 in the entries of $\mathbf{A}$ and $\mathbf{A}'$, thus providing a much simpler expression of the rigidity constraint than the one obtained by the purely algebraic methods described at the beginning of this section.

In section 12.5.5 we give another interpretation of the Kruppa equations in terms of the quadric dual to the absolute conic.

Equivalence between the Kruppa and the Huang-Faugeras equations It would be reassuring to show that the two formulations, the algebraic and the geometric, are equivalent. This is our next task. We have to show that the set of Kruppa equations (12.20) is equivalent to the constraint induced by the second equation (12.14). This is something which is a priori not intuitive, since we just saw that there are *two* Kruppa equations.

Proposition 12.9 *The two Kruppa equations are equivalent to the constraint obtained from the second Huang-Faugeras equations.*

Proof : Let us make a change of retinal coordinate system in each of the two retinal planes, so that the new Fundamental matrix is diagonalised. To achieve this, we use the singular value decomposition : there exists two orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ such that $\mathbf{F} = \mathbf{U}\mathrm{diag}(\lambda, \mu, 0)\mathbf{V}^T$. where $\lambda \neq 0$ and $\mu \neq 0$ since we know that a Fundamental matrix must be of rank two.

We use matrix $\mathbf{V}$ to change retinal coordinates in the first retina and matrix $\mathbf{U}$ to change retinal coordinates in the second retina, the new intrinsic parameters matrices are $\mathbf{A}_n = \mathbf{AV}$ and $\mathbf{A'}_n = \mathbf{A'U}$ in the first and second retina, respectively. Furthermore, in the new pixel coordinate systems we have $\mathbf{e} = \mathbf{e}' = (0, 0, 1)^T$.

The Kruppa equations are then easily shown to be given by:

$$K_{22} : K_{11} : -K_{12} = \lambda^2 K'_{11} : \mu^2 K'_{22} : \lambda\mu K'_{12},$$

which yields:

$$Eq_1 = \mu(\lambda K_{11} K'_{12} + \mu K_{12} K'_{22}) \qquad Eq_2 = \lambda(\lambda K_{12} K'_{11} + \mu K_{22} K'_{12})$$
$$Eq_3 = \mu^2 K_{22} K'_{22} - \lambda^2 K_{11} K'_{11}$$

Let us now express the condition $f(\mathbf{E}) = 0$. Since: $\mathbf{E} \simeq \mathbf{A}'^T \mathbf{FA}$, some algebra (done partially using the symbolic computation program MAPLE), leads to:

$$f(\mathbf{E}) \simeq -\frac{1}{2}((\lambda^2 K_{11} K'_{11} - \mu^2 K_{22} K'_{22})^2 + 2\lambda\mu(\lambda K_{12} K'_{11} + \mu K_{22} K'_{12})(\lambda K_{11} K'_{12} + \mu K_{12} K'_{22})),$$

so that

$$f(\mathbf{E}) \simeq -\frac{1}{2}(Eq_3^2 + 2Eq_1 Eq_2)$$

It is clear that if the Kruppa equations are satisfied, then $f(\mathbf{E}) = 0$.

Let us now prove the converse.

We notice that:

$$K_{22} Eq_1 - K_1 Eq_2 = K_{12} Eq_3 \tag{12.22}$$

In the case where $K_{12} \neq 0$, the previous equation can be rewritten, using (12.22):

$$(K_{22} Eq_1 - K_{11} Eq_2)^2 + 2Eq_1 Eq_2 K_{12}^2 = 0 \tag{12.23}$$

Thus, expanding:

$$K_{22}^2 Eq_1^2 + K_{11}^2 Eq_2^2 = 2Eq_1 Eq_2 (K_{11} K_{22} - K_{12}^2)(K_{11} K_{22} - K_{12}^2) \tag{12.24}$$

Because $\mathbf{K} = \mathbf{A}_n\mathbf{A}_n^T$ and $\mathbf{K}' = \mathbf{A}'_n{\mathbf{A}_n}'^T$, if we denote by $\mathbf{l}_i$, $i = 1, 2, 3$ and $\mathbf{l}'_i$, $i = 1, 2, 3$ the row vectors of $\mathbf{A}_n$ and $\mathbf{A}'_n$:

$$\begin{aligned} K_{12} &= \mathbf{l}_1^T\mathbf{l}_2 \\ K_{22} &= \|\mathbf{l}_2\|^2 \\ K_{11} &= \|\mathbf{l}_1\|^2 \end{aligned} \tag{12.25}$$

According to the definitions (12.26) of K_{12}, K_{11}, K_{22}, the Schwartz inequality implies that $K_{11}K_{22} - K_{12}^2$ is superior or equal to zero. If it is zero, one obtains from (12.24) $K_{11}E_1 = K_{22}E_2 = 0$. Since $K_{22}K_{11} = K_{12}^2 \neq 0$, it follows that $E_1 = E_2 = 0$. If it is strictly positive, then $E_1E_2 \geq 0$. The equation (12.23) is the sum of two positive terms, thus they have to be simultaneously zero, thus $E_1E_2 = 0$ and $E_3 = 0$.

The case which remains is $K_{12} = 0$. The Kruppa equations take then the simple form:

$$\lambda\mu K_{11}K'_{12} = \lambda\mu K_{22}K'_{12} = \mu^2K_{22}K'_{22} - \lambda^2K_{11}K'_{11} = 0$$

which is equivalent to:

$$\begin{cases} K'_{12} = 0 \\ \lambda^2K_{11}K'_{11} - \mu^2K_{22}K'_{22} = 0 \end{cases} \quad \text{or} \quad \begin{cases} K'_{12} \neq 0 \\ K_{22} = K_{11} = 0 \end{cases}$$

and to:

$$f(\mathbf{E}) \simeq 2\lambda^2\mu^2K_{22}K_{11}K'^2_{12} + (\lambda^2K_{11}K'_{11} - \mu^2K_{22}K'_{22})^2 = 0$$

□

As a consequence of this proof, we obtain the following formulas which give the Kruppa equations as a function of the singular value decomposition of the Fundamental matrix:

Proposition 12.10 *The Kruppa equations derived from the SVD of the Fundamental matrix* $\mathbf{F} = \mathbf{U}diag(\lambda, \mu, 0)\mathbf{V}^T$*, where* $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ *and* $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ *are two orthogonal matrices, are:*

$$\frac{\lambda^2\mathbf{v}_1^T\mathbf{K}\mathbf{v}_1}{\mathbf{u}_2^T\mathbf{K}\mathbf{u}_2} = \frac{\mu^2\mathbf{v}_2^T\mathbf{K}\mathbf{v}_2}{\mathbf{u}_1^T\mathbf{K}\mathbf{u}_1} = -\frac{\lambda\mu\mathbf{v}_1^T\mathbf{K}\mathbf{v}_2}{\mathbf{u}_1^T\mathbf{K}\mathbf{u}_2}$$

It is interesting to note that in the calibrated case, $\mathbf{K} = \mathbf{I}_3$, and therefore those equations reduce to $\lambda^2 = \mu^2$, in accordance with the condition on the Essential matrix in Eq. (6.34).

12.3.3 Using two displacements of a camera: a method for self-calibration

The idea behind the method is to use the Kruppa equations in the case of a single camera that we move in a static scene while keeping its intrinsic parameters fixed. In that case $\mathbf{A} = \mathbf{A}'$, $\mathbf{K} = \mathbf{K}'$, and each pair of views provides potentially two equations that constrain the intrinsic parameters. Intuitively, if sufficiently many different displacements are made, one might hope that one will be able to recover all intrinsic parameters.

How many displacements or images are necessary? Each displacement yields two independent algebraic equations. We have in general five coefficients to estimate and therefore three displacements are necessary. In the case where some intrinsic parameters have known values, e.g. 0 skew, a smaller number of displacements may be sufficient. For example, in the case of 0 skew, we have the additional constraint (5.37).

Given three views taken from three different viewpoints, there are three displacements, noted D_{12}, D_{23}, and D_{13}. One could worry about the fact that since the third displacement is a composition of the first two, $D_{13} = D_{23}D_{12}$, the two equations provided by D_{13} are algebraically dependent on the four equations provided by D_{12} and D_{23}, thus resulting in an under-constrained system. One way to see that it is not the case is to count unknowns. Two Fundamental matrices depend only upon 14 parameters (see chapter 6). On the other hand, the Euclidean information which is recovered by self-calibration, consists of the three displacements, i.e. three rotations and three translations defined up to a common scale factor, and the 5 intrinsic parameters. The displacements depend on 11 parameters: 3 for each of the rotations $\mathbf{R}_{21}$ and $\mathbf{R}_{32}$, 2 for each of the directions of translations $\mathbf{t}_{12}$ and $\mathbf{t}_{23}$, one for the ratio of the norms of the translations. The total is 16 parameters, thus the information is not entirely contained in the first two Fundamental matrices. The two missing parameters are actually recovered thanks to the two additional Kruppa equations provided by the third Fundamental matrix.

We give now a simple numerical example to show that in the general case the equations are independent. The first two displacements are:

$$\mathbf{R}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{t}_{21} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{R}_{32} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{t}_{32} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

The displacement obtained by composition of D_1 and D_2, expressed in the coordinate system of the first camera, is:

$$\mathbf{R}_{31} = \mathbf{R}_{21}\mathbf{R}_{32} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{t}_{31} = \mathbf{R}_{21}\mathbf{t}_{32} + \mathbf{t}_{21} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

If we take as intrinsic parameter matrix $\mathbf{A}$ the identity matrix, the Fundamental matrices are identical to the essential matrices. Setting $K_{33} = 1$, the six Kruppa equations are:

$$\begin{aligned}
Eq_1 \;&=\; 4\,K_{11} - 2\,K_{13}K_{11} - 4\,K_{23}K_{11} - 4\,K_{22}K_{11} + 6\,K_{12}K_{11} + 4\,K_{13}K_{22} - 2\,K_{23}K_{22} \\
&\quad + 2\,K_{12}K_{22} - 4\,K_{12}^2 - 4\,K_{12} + 4\,K_{23}K_{12} - 4\,K_{13}^2 + 2\,K_{13} + 4\,K_{23}K_{13} - 2\,K_{23} \\
Eq_1' \;&=\; 3\,K_{11} - 8\,K_{23}K_{11} - 3\,K_{22}K_{11} + K_{22}^2 + 8\,K_{13}K_{22} - 4\,K_{23}K_{22} - 2\,K_{12}K_{22} \\
&\quad - 8\,K_{12} + 8\,K_{23}K_{12} + 2\,K_{13} + 8\,K_{23}K_{13} - 4\,K_{23} - 1 \\
Eq_2 \;&=\; 4\,K_{13}K_{11} + 2\,K_{12}K_{11} + 4\,K_{23}K_{22} + 2\,K_{12}K_{22} + 16\,K_{12} + 8\,K_{13}K_{12} + 8\,K_{23}K_{12} \\
&\quad + 16\,K_{13}^2 + 16\,K_{13} + 16\,K_{23}^2 + 16\,K_{23} \\
Eq_2' \;&=\; -K_{11}^2 - 4\,K_{11} - 4\,K_{13}K_{11} + K_{22}^2 + 4\,K_{22} + 4\ ,K_{23}K_{22} \\
Eq_3 \;&=\; -18\,K_{11}^2 + 60\,K_{13}K_{11} - 52\,K_{23}K_{11} + 12\,K_{12}K_{11} + 18\,K_{22} - 6\,K_{13}K_{22} - 6\,K_{23}K_{22} \\
&\quad + 2\,K_{12}K_{22} + 12\,K_{12}^2 + 54\,K_{12} - 36\,K_{23}K_{12} - 36\,K_{13}^2 + 54\,K_{13} - 12\,K_{23}K_{13} + 18\,K_{23} \\
Eq_3' \;&=\; -9\,K_{11}^2 - K_{11} + 60\,K_{13}K_{11} - 54\,K_{23}K_{11} + 9\,K_{22}K_{11} + K_{22}^2 + 9\,K_{22} \\
&\quad - 6\,K_{23}K_{22} + 6\,K_{12}K_{22} + 54\,K_{12} - 36\,K_{23}K_{12} - 36\,K_{13}^2 + 60\,K_{13} - 9
\end{aligned}$$

A solution of the system of equations Eq_1, Eq_1', Eq_2, Eq_2' obtained from the displacements $\mathbf{D}_1$ and $\mathbf{D}_2$ is:

$$K_{11} = 0 \quad K_{12} = -1 \quad K_{13} = \frac{1}{2} \quad K_{22} = -4 \quad K_{23} = 0$$

Substituting these values into the equations obtained from $\mathbf{D}_3$ yields: $Eq_3 = -76$, $Eq_3' = -38$, thus we have verified that these equations are independent from the previous ones, i.e that they cannot be written as polynomials in Eq_1, Eq_1', Eq_2, Eq_2'.

Degenerate cases There are many cases where the Kruppa equations are degenerate, i.e. do not provide enough information for recovering the internal parameters. In this section we discuss only a few of the simplest cases. Section 12.7 provides a more detailed analysis. There are situations when one displacement is such that the Kruppa equations do not provide any constraint at all: they include pure translation, pure rotation, and affine viewing situation.

Pure translation In the case where the displacement is a pure translation (the rotation is the identity $\mathbf{R} = \mathbf{I}_3$), it can be seen from equation (12.13) that the Fundamental matrix is antisymmetric. Since $\mathbf{E} = [\mathbf{t}]_\times$ and $\mathbf{A} = \mathbf{A}'$, we have $\mathbf{F} \simeq \mathbf{A}^{-T}[\mathbf{t}]_\times\mathbf{A}$ which is an antisymmetric matrix. If $\mathbf{F}$ is antisymmetric, $\mathbf{e} \simeq \mathbf{e}'$ From (6.16), we conclude that

$$\forall \mathbf{y}, \mathbf{F}\mathbf{y} \simeq \mathbf{e} \times \mathbf{y}$$

The equations (12.18) and (12.19) are equivalent, and thus the Kruppa equations reduce to tautologies. A geometric interpretation is that since the two tangents to ω in the first image *are* the tangents to ω in the second image, no further constraint is put on ω by considering a second image.

Pure rotation In the case of 0 translation, the Fundamental matrix is the null matrix and there are no epipolar constraints and thus no Kruppa equations. The displacement is a pure rotation whose axis contains the optical center of the camera. Section 12.2 is dedicated to this case.

Affine viewing If the viewing geometry can be described with the affine projection model, then it can be verified by substitution of the form of the affine Fundamental matrix given in Equation (6.39) into the Kruppa equations given in Equation (12.17) that they reduce to the form $K_{33}f(a, b, c, d)$, which is never or always zero. Let see what happens to the geometric constraint from which the Kruppa equations are derived.

- The plane at infinity Π_∞ is mapped into the line at infinity l_∞ by an affine projection. The absolute conic is mapped into two points I, J of l_∞ in the first image and two points I', J' of l_∞ in the second image.
- Since the epipolar lines are parallel, they intersect l_∞ at a single point d in the first image, d' in the second image.
- The tangency condition reduces to the fact that d is either I or J and d' is either I' or J'.

12.4 Computing the intrinsic parameters using the Kruppa equations

This section is devoted to a first family of techniques for solving the problem of recovering the Euclidean stratum from the projective stratum, represented by a collection of Fundamental matrices. They do this by solving one way or another the Kruppa equations to recover the intrinsic parameters. They then exploit the relation (6.33) between the fundamental matrix and the essential matrix to recover the essential matrixes from which the relative rotations and the directions of the translations can be recovered, see section 6.3.3. Note that the ratio of the lengths of any of the translation vectors can be computed using propositions 11.6, 11.7 and 11.8, or as explained in section 12.6.1.

These methods separate entirely the estimation of the intrinsic parameters from the estimation of the motion parameters. Sec. 12.4.1 details formulas to recover the focal lengths of two cameras (certainly the most important intrinsic parameter) from a single Fundamental matrix. Sec. 12.4.2 discusses methods which take advantage of the polynomial structure of the equations to solve from three views with constant intrinsic parameters. Sec. 12.4.3 presents an optimization based approach to solve the same equations, which is efficient to integrate multiple views. It is an interesting application of the methods, described in chapter 7, for estimating the uncertainty of the fundamental matrix.

12.4.1 Recovering the focal lengths for two views

One Fundamental matrix yields two Kruppa equations, therefore we should be able to recover two intrinsic parameters assuming that the rest of them is known. We consider cameras which have zero-skew, unit aspect ratio (Sec. 5.3.1), and known principal points $\mathbf{c}$ and $\mathbf{c}'$:

Proposition 12.11 (Bougnoux) *For a pair of cameras with Fundamental matrix $\mathbf{F}$, epipole $\mathbf{e}'$, and intrinsic parameters:*

$$\mathbf{A} = \begin{bmatrix} f & 0 & u_0 \\ 0 & f & v_0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad \mathbf{A}' = \begin{bmatrix} f' & 0 & u'_0 \\ 0 & f' & v'_0 \\ 0 & 0 & 1 \end{bmatrix}$$

the focal of the first camera is given by:

$$f = \sqrt{-\frac{\mathbf{c}'^T [\mathbf{e}']_\times \tilde{\mathbf{I}}\mathbf{F}\mathbf{c}(\mathbf{c}^T\mathbf{F}^T\mathbf{c}')}{\mathbf{c}'^T [\mathbf{e}']_\times \tilde{\mathbf{I}}\mathbf{F}\tilde{\mathbf{I}}\mathbf{F}^T\mathbf{c}'}} \quad \textit{where } \tilde{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (12.26)$$

and $\mathbf{c} = [u_0, v_0, 1]^T$ and $\mathbf{c}' = [u'_0, v'_0, 1]^T$ are the principal points.

Proof : Under our assumptions, the Kruppa matrix of the first camera can be written:

$$\mathbf{K} = \mathbf{A}\mathbf{A}^T = \begin{bmatrix} f^2 + u_0^2 & u_0 v_0 & u_0 \\ u_0 v_0 & f^2 + v_0^2 & v_0 \\ u_0 & v_0 & 1 \end{bmatrix} = f^2 \tilde{\mathbf{I}} + \mathbf{c}\mathbf{c}^T$$

Let's note $\mathbf{l} = \mathbf{F}\mathbf{c}$, $\mathbf{m} = [\mathbf{e}']_\times \mathbf{c}'$, $\tilde{\mathbf{F}} = \mathbf{F}\tilde{\mathbf{I}}\mathbf{F}^T$ and

$$\tilde{\mathbf{E}}' = [\mathbf{e}']_\times \tilde{\mathbf{I}} [\mathbf{e}']_\times^{\mathbf{T}} = \begin{bmatrix} {e'}_3^2 & 0 & -{e'}_1 {e'}_3 \\ 0 & {e'}_3^2 & -{e'}_2 {e'}_3 \\ -{e'}_1 {e'}_3 & -{e'}_2 {e'}_3 & {e'}_1^2 + {e'}_2^2 \end{bmatrix} \tag{12.27}$$

The Kruppa equations (12.17) can be rewritten with the proportionnality factor x:

$$f^2 \tilde{\mathbf{F}} + \mathbf{l}\mathbf{l}^T = x({f'}^2 \tilde{\mathbf{E}}' + \mathbf{m}\mathbf{m}^T) \tag{12.28}$$

Using $\tilde{E'}_{12} = 0$, if $m_1 m_2 = 0$, we conclude from that equation that:

$$f^2 \tilde{F}_{12} + l_1 l_2 = 0$$

otherwise, we obtain the scale factor $x = \frac{f^2 \tilde{F}_{12} + l_1 l_2}{m_1 m_2}$, which we substitute back into Eq. (12.28):

$$f^2 {f'}^2 \tilde{F}_{12} \tilde{E'} + f^2 (\tilde{F}_{12} \mathbf{m}\mathbf{m}^T - m_1 m_2 \tilde{F}) + {f'}^2 l_1 l_2 \tilde{\mathbf{E}}' + l_1 l_2 \mathbf{m}\mathbf{m}^T - m_1 m_2 \mathbf{l}\mathbf{l}^T = 0$$

This is a polynomial equation of degree 2 in f^2 and ${f'}^2$ with matrix coefficients. We can rewrite it as $f^2 {f'}^2 \mathbf{A} + f^2 \mathbf{B} + {f'}^2 \mathbf{C} + \mathbf{D} = \mathbf{0}_{3\times 3}$ with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ four 3×3 symmetric matrices which depend only on the epipolar geometry and principal points. Let's call EQ those 6 equations. EQ_{12} was already used for the parameter x.

Now from $\tilde{E'}_{11} = \tilde{E'}_{22}$, we conclude that we must have $EQ_{11} = EQ_{22}$, which means: $A_{11} = A_{22}$, $C_{11} = C_{22}$ and:

$$(B_{11} - B_{22}) f^2 + D_{11} - D_{22} = 0 \tag{12.29}$$

we obtain with some algebra:

$$\begin{aligned} B_{11} &= \tilde{F}_{12} m_1^2 - \tilde{F}_{11} m_1 m_2 &&= -m_1 \begin{bmatrix} F_{11} & F_{12} & 0 \end{bmatrix} \mathbf{F}^T \mathbf{c}' e'_3 \\ D_{11} &= l_1 l_2 m_1^2 - m_1 m_2 l_1^2 &&= -l_1 m_1 (\mathbf{c}^T \mathbf{F} \mathbf{c}') e'_3 \end{aligned}$$

and

$$\begin{aligned} B_{22} &= \tilde{F}_{12} m_2^2 - \tilde{F}_{22} m_1 m_2 &&= m_2 \begin{bmatrix} F_{21} & F_{22} & 0 \end{bmatrix} \mathbf{F}^T \mathbf{c}' e'_3 \\ D_{22} &= l_1 l_2 m_2^2 - m_1 m_2 l_2^2 &&= D_{22} = -l_2 m_2 (\mathbf{c}^T \mathbf{F} \mathbf{c}') e'_3 \end{aligned}$$

We remark that $e'_1 EQ_{11} = -e'_3 EQ_{13}$ and $e'_2 EQ_{21} = -e'_3 EQ_{23}$. So, as EQ_{13} and EQ_{23} should be verified we can always simplify by m_3 and write Eq. (12.29) as:

$$\mathbf{m}^T \tilde{\mathbf{I}}(\tilde{\mathbf{F}} f^2 + \mathbf{F}\mathbf{c}\mathbf{c}^T \mathbf{F}^T)\mathbf{c}' = 0$$

from which the announced formula follows. □

It can be noted that this formula gives a null focal (a degenerate value) if $\mathbf{c}^T \mathbf{F}^T \mathbf{c}' = 0$, or in other words, the two principal points are in epipolar correspondence. This actually means that the two optical axis are intersecting at a common point, a situation which will be studied in more detail in Sec. 12.7. In the same spirit as the previous calculations, it is also possible to solve for the scale factors in each direction α_u and α_v if they are supposed constant in the two views. The Kruppa equations yield a system of two polynomials of degree two with two unknowns which has an analytical solution.

12.4.2 Solving the Kruppa equations for three views

We have seen that three displacements yield six equations in the entries of the Kruppa matrix **K**. The equations are homogeneous, so the solution is determined only up to a scale factor. In effect there are five unknowns. Trying to solve the over-determined problem with numerical methods usually fails, so five equations are picked from the six and solved first. As the equations are each of degree two, the number of solutions in the general case is $32 = 2^5$. The remaining equation could just be used to discard the spurious solutions, but this ignores the uncertainty on the coefficients and it is better to exploit the redundancy of information to obtain a more robust algorithm, as well as a gross estimate of the variance of the solutions.

How do we solve a system of 5 polynomial equations of degree 2 in 5 variables? Providing an initial guess and using an iterative numerical method will generally not find all the solutions: many of the starting points will yield trajectories that do not converge and many other trajectories will converge to the same solution. However it is not acceptable to miss solutions, since there is only one correct one amongst the 32.

Recently developed methods in numerical continuation can reliably compute all solutions to polynomial systems. These methods have been improved over a decade to provide reliable solutions to kinematics problems. Some details of these improvements are explained in [403], which is a detailed tutorial presentation. The solution of a system of nonlinear equations by numerical continuation is suggested by the idea that small changes in the parameters of the system usually produce small changes in the solutions. Suppose the solutions to problem A (the initial system) are known and solutions to problem B (the target system) are required. Solutions are tracked as the parameters of the system are slowly changed from those of A to those of B. Although for a general nonlinear system numerous difficulties can arise, such as divergence or bifurcation of a solution path, for a polynomial system all such difficulties can be avoided.

Details and experimental results can be found in [223]. The main advantage of the method is that no initialization is needed. If the points are measured with a good precision, the results can be sufficiently precise. Another advantage is that it is easy to assess the success or failure of the algorithm. However there are several drawbacks:

- the method is suitable only for the case of the minimum number of displacements, as it is difficult to use all the $\binom{n}{5}$ constraints provided by a sequence of n views without increasing considerably the amount of computations,

- it is difficult to take into account uncertainty for the input (fundamental matrices) as well as producing estimates of the uncertainty for the output (camera parameters),

- the computational cost of solving the polynomial system is relatively high,

- it is not possible to express the constraints of positivity of $\mathbf{K}$ when solving the systems, since continuation works in the complex plane. Thus with noisy data, it can happen that no acceptable solution are found.

It is easy to enforce the zero-skew constraint $\gamma = 0$ within this framework, since it is also a polynomial of degree two in the entries of the Kruppa matrix (equation (5.37)). Two displacements and this constraint generate a system of five equations, and are therefore sufficient to find a solution.

12.4.3 Non-linear optimization to accumulate the Kruppa equations for $n > 3$ views

Suppose we have n views with constant intrinsic parameters, and estimated $N \leq \binom{n}{2}$ fundamental matrixes, noted $\mathbf{F}_i$, between pairs of views. For each fundamental matrix, we consider two Kruppa equations (12.21). Those equations are polynomials of degree two in the entries of the Kruppa matrix $\mathbf{K}$

with coefficients that are functions of the entries of $\mathbf{F}_i$, we note $Eq_1(\mathbf{F}_i, \mathbf{K})$ and $Eq_2(\mathbf{F}_i, \mathbf{K})$ those two equations (see section 12.3.2).

For each matrix $\mathbf{K}$, we can compute, using the techniques described in section 7.7, the variances $\sigma_{Eq_j}(\mathbf{F}_i, \mathbf{K})$, $j = 1, 2$ of the two Kruppa equations as:

$$\sigma^2_{Eq_j} = (\frac{\partial Eq_j}{\partial \mathbf{F}_i})^T \mathbf{\Lambda}_{\mathbf{F}_i} \frac{\partial Eq_j}{\partial \mathbf{F}_i}$$

One then minimizes with respect to the five parameters of $\mathbf{K}$ the following error criterion:

$$\sum_{i=1}^{N} \frac{Eq_1(\mathbf{F}_i, \mathbf{K})^2}{\sigma_{Eq_1}(\mathbf{F}_i, \mathbf{K})} + \frac{Eq_2(\mathbf{F}_i, \mathbf{K})^2}{\sigma_{Eq_2}(\mathbf{F}_i, \mathbf{K})} \qquad (12.30)$$

To enforce the fact that $\mathbf{K} = \mathbf{A}\mathbf{A}^T$ has to be a positive matrix, the minimization is done with the parameters of the intrinsic parameters matrix $\mathbf{A}$ defined in Eq. (5.30). Several variants are possible depending on the simplification on $\mathbf{A}$ chosen (see Sec. 5.3.1), satisfactory results being obtained with the three parameter model (zero-skew, unit aspect ratio).

There are several ways of finding a starting point for the minimization of (12.30). One way is to choose two (zero skew) or three views among the n and to solve the corresponding Kruppa equations by the method described in the previous section. Another method, which does not require iterative computations and can be used robustly with several views, is described by Zeller [425]:

- For each fundamental matrix $\mathbf{F}_i$, take $\gamma = 0$, the principal point at the image center, and solve analytically the two Kruppa equations for α_i and β_i.

- The initial values for α and β are taken as the median values of the α_i and β_i.

An example of results obtained with three different synthetic sequences is shown in Fig. 12.7. In each sequence, 100 points, which are confined to the inside of a cube, are observed from the ten viewpoints which are indicated by their camera coordinate system and the optical axis. The noise level is 1.0 pixel (in a 512×512 image).

In the first sequence, the viewpoints are well-distributed, roughly on a sphere centered at the middle of the cube and the optical axes are roughly going through the center of the sphere: this is a good approximation to the critical sequence described in section 12.7.2. In the second sequence, the views are almost coplanar: this is one of the intrinsic critical sequences described in section 12.7.1. But they are also roughly on the same sphere as in the first sequence. The third sequence is a case of a critical sequence where the rotation axes are identical.

The second row of figure 12.7 shows the different values of α_i and β_i obtained by the analytical method described above. It can be seen that for all three sequences the pairs $(\alpha_i,\ \beta_i)$ roughly fall on a line whose slope is close to the correct aspect ratio. This confirms the analysis of section 12.7.2. It has been found that the aspect ratio computed by this method is quite insensitive to the choice of (u_0, v_0).

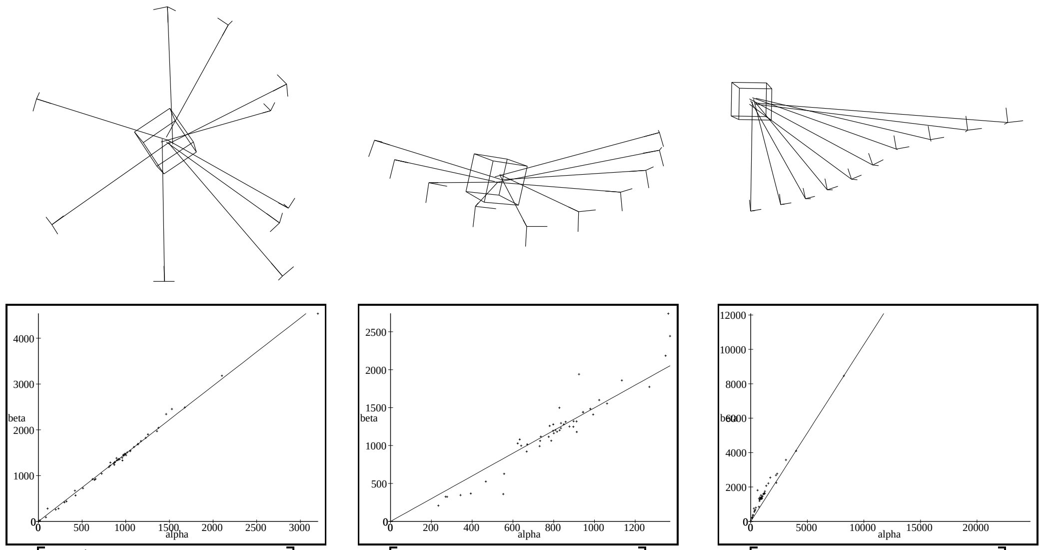
12.5 Computing the canonic transformation parameters

All the methods that are described in this section start from the projective stratum computed as explained in section 11.3, and represented by the projection matrices. They upgrade the representation from projective to affine and Euclidean by computing directly the projection matrices. The main idea is to notice that, once the projective projection matrices have been obtained, there only remains to find one collineation of space to turn them into affine or Euclidean projection matrices. We start in Sec. 12.5.1 with the affine camera case, where we start already from affine. In the perpective case, as explained in Sec. 12.5.2, the collineation can very simply be parametrized by 8 parameters: 5 parameters for the intrinsic parameters and 3 for the plane at infinity. We then examine different families of methods. The first family produce an affine representation (Sec. 12.5.3), the second family relies on an intermediate affine stage (Sec. 12.5.4), and the third family treats the two stages uniformly (Sec. 12.5.5). A further advantage of these methods approach is that they extend naturally to the case of varying internal parameters, discussed in Sec. 12.5.6.

12.5.1 The affine camera case

If the camera performs a parallel projection (affine camera, see chapter 5), the ambiguity in reconstruction is already at most affine. We examine now how to recover the intrinsic parameters of the affine camera so that Euclidean information can be obtained. We have seen in section 11.3.3.2 how to recover the 2×3 projection matrices $\mathbf{P}_i^A, i = 1, \cdots, n$ of a set of affine cameras up to an arbitrary affine transformation of the space.

We have given in chapter 5, section 5.4.3, several definitions of the intrin-



init	$\begin{bmatrix} 917.3 & 0 & 256 \\ 0 & 1354.7 & 256 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 816.9 & 0 & 256 \\ 0 & 1203.3 & 256 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 944.8 & 0 & 256 \\ 0 & 1372.6 & 256 \\ 0 & 0 & 1 \end{bmatrix}$
5 param	$\begin{bmatrix} 897.4 & -0.23 & 246.8 \\ 0 & 1325.8 & 259.1 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 950.8 & -3.21 & 249.9 \\ 0 & 1402.7 & 224.3 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1163.5 & 3.82 & 132.1 \\ 0 & 1683.7 & 4.1 \\ 0 & 0 & 1 \end{bmatrix}$
3 param	$\begin{bmatrix} 905.6 & 0 & 246.1 \\ 0 & 1340.6 & 259.6 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 914.4 & 0 & 253.2 \\ 0 & 1369.1 & 223.2 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2436.1 & 0 & -227.7 \\ 0 & 2504.8 & -1110.5 \\ 0 & 0 & 1 \end{bmatrix}$
exact		$\begin{bmatrix} 914.6 & -0.009 & 249 \\ 0 & 1352.1 & 259 \\ 0 & 0 & 1 \end{bmatrix}$	

Figure 12.7: Results obtained by minimizing an error function based on the Kruppa equations weighted by their uncertainty. The first row shows the configuration of the ten views. The second row shows the distribution of the pairs (α_i, β_i), see text. The third row shows the initial matrixes $\mathbf{A}$. The next two rows show the values obtained after optimization respectively with respect to the five parameters of $\mathbf{A}$, or only with three parameters ($\gamma = 0$, $\beta = k_0\alpha$).

sic parameters of an affine camera. They all come down to the decomposition of the Euclidean matrices $\mathbf{P}_i^E$ as:

$$\mathbf{P}_i^E = \mathbf{A}_i \mathbf{R}_i$$

where $\mathbf{A}_i$ is the 2×3 matrix of the intrinsic parameters and $\mathbf{R}_i$ is a 3×3 matrix of rotation. The self-calibration method consists in finding an invertible 3×3 matrix $\mathbf{T}$ corresponding to an affine change of coordinates such that:

$$\mathbf{P}_i^A \mathbf{T} = \mathbf{P}_i^E = \mathbf{A}_i \mathbf{R}_i \tag{12.31}$$

The unknowns are $\mathbf{T}$, $\mathbf{A}_i$ and $\mathbf{R}_i$. In order to succeed, one needs to make some extra assumptions. One reasonable such assumption is that the matrix of intrinsic parameters is "almost" constant:

$$\mathbf{A}_i = \lambda_i \mathbf{A}$$

where λ_i varies from one camera to the next. We can eliminate the rotation matrix $\mathbf{R}_i$ in (12.31) by transposing and multiplying:

$$\mathbf{P}_i^A \mathbf{T} \mathbf{T}^T \mathbf{P}_i^{AT} = \lambda_i^2 \mathbf{A} \mathbf{A}^T$$

Let $\mathbf{X} = \mathbf{T}\mathbf{T}^T$, $\mathbf{q}_i$ and $\mathbf{s}_i$ be the two row vectors of $\mathbf{P}_i^A$. The previous equation can be rewritten

$$\begin{bmatrix} \mathbf{q}_i^T \mathbf{X} \mathbf{q}_i & \mathbf{c}_i^T \mathbf{X} \mathbf{s}_i \\ \mathbf{q}_i^T \mathbf{X} \mathbf{s}_i & \mathbf{s}_i^T \mathbf{X} \mathbf{s}_i \end{bmatrix} = \lambda_i^2 \mathbf{A} \mathbf{A}^T \tag{12.32}$$

Writing the two proportionality equations for each pair of consecutive views to eliminate the unknown scale factor gives $2(n-1)$ relations which are quadratic in the entries of $\mathbf{X}$. Since $\mathbf{X}$ is symmetric and defined up to a scale factor, it has five unknowns. Therefore, in the case of the general affine camera, four views are necessary. If we suppose further that the unknown scale factor λ_i is constant, then we can write three equalities, which are linear in the entries of $\mathbf{X}$, so that only three views are necessary.

12.5.2 The general formulation in the perspective case

Starting from a set of projective projection matrixes $\boldsymbol{\mathcal{P}}_i$, the core of all those methods is to find a collineation $\boldsymbol{\mathcal{H}}$ of $\mathbb{P}^3$ which transforms these matrices into their canonical Euclidean forms:

$$\boldsymbol{\mathcal{P}}_i \boldsymbol{\mathcal{H}} \simeq \mathbf{A}[\mathbf{R}_i\, \mathbf{t}_i] \qquad 1 \le i \le n \tag{12.33}$$

where we can assume that $\mathbf{R}_1 = \mathbf{I}_3$ and $\mathbf{t}_1 = \mathbf{0}$ (see section 11.4).

With the notations of chapter 8, we have (equation (8.18)):

$$\boldsymbol{\mathcal{H}} \equiv \mathbf{Q}_{\mathcal{E}}^{\mathcal{B}_0} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{r}_\infty^T \mathbf{A} & \mu \end{bmatrix}, \tag{12.34}$$

and write the projective projection matrices as:

$$\boldsymbol{\mathcal{P}}_1 = [\mathbf{I}_3\, \mathbf{0}], \qquad \boldsymbol{\mathcal{P}}_i = [\mathbf{P}_i\, \mathbf{p}_i],\ 2 \le i \le m$$

The equations (12.33) are equivalent to:

$$\begin{aligned} (\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)\mathbf{A} &\simeq \mathbf{A}\mathbf{R}_i \\ \mu \mathbf{p}_i &\simeq \mathbf{A}\mathbf{t}_i \end{aligned} \tag{12.35}$$

These $2(n-1)$ equations form the basis for all the self-calibration algorithms that we describe next.

12.5.3 From projective to affine

The method of Pollefeys and Van Gool [272] relies on the assumption of constant intrinsic parameters to recover the affine stratum.

Let us start with equation (12.35) and multiply both sides on the right with the matrix $\mathbf{A}^{-1}$:

$$\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T \simeq \mathbf{A}\mathbf{R}_i\mathbf{A}^{-1}$$

The righthand side is recognized as being the infinity morphism between views 1 and i (proposition 6.25). This induces a constraint on the lefthand side: it is proportional to a matrix which is similar to a rotation matrix. Since the eigenvalues of a rotation matrix are of unit magnitude, the eigenvalues of the matrix $\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T$ must have the same magnitudes, as mentioned in proposition 6.26. This can be written as a fourth degree polynomial in the coordinates of $\mathbf{r}_\infty$. If the characteristic polynomial is

$$\det(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T - \lambda \mathbf{I}_3) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

a necessary condition is $a_3 a_1^3 = a_2^3 a_0$. Each view, except the first provides one such constraint on $\mathbf{r}_\infty$ and therefore four views are in general sufficient to obtain a finite number of solutions.

Once the affine stratum has been recovered, the Kruppa matrix can be recovered from the matrixes $\mathbf{H}_{\infty 1i}$ as described in section 12.2.

12.5.4 Searching for the plane at infinity

As proposed for example by Hartley [134, 142], one can recover simultaneously the affine stratum (represented by the vector $\mathbf{r}_\infty$) and the Euclidean stratum (represented by the matrix of intrinsic parameters $\mathbf{A}$).

The main ideas are as follows. Given a value of $\mathbf{r}_\infty$, the infinity morphism from view 1 to view i can be computed for all i's:

$$\mathbf{H}_{\infty 1i} = \mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T$$

Next, the Kruppa matrix can be obtained from the set of $\mathbf{H}_{\infty 1i}$ using the methods described in section 12.2. If this matrix is definite, the matrix $\mathbf{A}$ can be obtained by Cholesky decomposition.

Consider now the QR decomposition of the matrix in the lefthand side of equation (12.35):

$$(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)\mathbf{A} = \mathbf{A}'_i \mathbf{R}'_i,$$

and notice that if one has the correct values for $\mathbf{r}_\infty$ and $\mathbf{A}$, one has $\mathbf{R}_i = \mathbf{R}'_i$ and $\mathbf{A}'_i \simeq \mathbf{A}$. It is therefore natural to minimize the following error function:

$$\sum_{i=2}^{n} \|\alpha_i \mathbf{A}^{-1}\mathbf{A}'_i - \mathbf{I}_3\|^2$$

with respect to $\mathbf{A}$ and $\mathbf{r}_\infty$. The numbers α_i are equal to $3\,\mathrm{sgn}(\det(\mathbf{A}^{-1}\mathbf{A}'_i))/\mathrm{tr}(\mathbf{A}^{-1}\mathbf{A}'_i)$ so that the sum of the squares of the three diagonal elements of the matrixes $\mathbf{A}^{-1}\mathbf{A}'_i$ is equal to 3.

There is no closed-form solution to this minimization problem. To obtain initialization values for $\mathbf{A}$ and $\mathbf{r}_\infty$, the idea is to try different values of $\mathbf{r}_\infty$, until a value of $\mathbf{A}$ can be computed, i.e. until the Kruppa matrix is definite. If it is not definite, then another value of $\mathbf{r}_\infty$ is tried.

Hartley describes an interesting way to restrain the possible values of $\mathbf{r}_\infty$, by considering the so-called *Cheiral inequalities*, resulting from the fact that all the reconstructed points have to lie in front of the cameras. To give a clear explanation of these inequalities would require a fair amount of details, so we refer the interested reader to [143, 133].

12.5.5 The absolute quadric

Heyden and Astrom's method Another possibility [158] to upgrade the representation from projective to Euclidean is to multiply each side of equation (12.35) by its transpose. This has the effect of eliminating the unknown rotation matrixes $\mathbf{R}_i$. One obtains the equations (called the Kruppa constraints in [158]):

$$(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)\mathbf{K}(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)^T \simeq \mathbf{K} \tag{12.36}$$

where $\mathbf{K}$ is the Kruppa matrix. This expression contains five equations, since the matrices are symmetric and defined up to a scale factor. Three camera matrices yield ten equations, which is enough to recover the eight unknowns.

If one introduces the $n-1$ scale factors $\lambda_2, \ldots, \lambda_n$, one is naturally led to the minimization of the following error function

$$\sum_{i=2}^{n} \|(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)\mathbf{K}(\mathbf{P}_i + \mathbf{p}_i \mathbf{r}_\infty^T)^T - \lambda_i \mathbf{K}\|^2$$

with respect to the $n-1$ scalars λ_i, the vector $\mathbf{r}_\infty$, and the definite symmetric matrix $\mathbf{K}$. Let us give an interpretation of this method in terms of the absolute quadric dual to the absolute conic (see chapter 3, section 3.6.3).

Trigg's method Suppose, as usual, that we have recovered the projective stratum and that the projection matrixes are given in a canonical projective coordinate frame $\mathcal{B}_0$ by the matrixes $\boldsymbol{\mathcal{P}}_i$, $i = 1, \ldots, n$. We now apply proposition 3.23 and consider the quadric dual to the absolute conic Ω. In that projective coordinate frame, this quadric is represented by a 4×4 symmetric matrix $\boldsymbol{\mathcal{Q}}^*_{\mathcal{B}_0}$ of rank 3, i.e. as a degenerate quadric. In effect it is a quadric cone, the set of all planes tangent to Ω. The vertex of the cone is the plane at infinity. In an Euclidean frame, the absolute quadric is represented by the matrix (equation (3.22)):

$$\boldsymbol{\mathcal{Q}}^*_{\mathcal{E}} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3^T & 0 \end{bmatrix}$$

The absolute quadric encodes Euclidean structure in a way similar to the absolute conic. It is unvariant under Euclidean transformation. It also encodes affine structure since the plane at infinity is its vertex, determined as a null vector of its matrix.

Let us now consider one camera represented by its projection matrix $\boldsymbol{\mathcal{P}}$. Let l be a line in the retinal plane of our camera, $\mathbf{l}$ one of its coordinate vectors. The plane Π defined by l and the optical center C of the camera is represented by the vector $\boldsymbol{\mathcal{P}}^T\mathbf{l}$ (see Sec. 5.1.2) It is tangent to the absolute conic if and only if it belongs to the absolute quadric, i.e.:

$$\mathbf{l}^T \boldsymbol{\mathcal{P}} \boldsymbol{\mathcal{Q}}^*_{\mathcal{B}_0} \boldsymbol{\mathcal{P}}^T \mathbf{l} = 0$$

The 3×3 symmetric matrix $\mathcal{P}\mathcal{Q}^*_{\mathcal{B}_0}\mathcal{P}^T$ defines a conic in the plane dual to the retinal plane of the camera which is of course nothing but our old friend ω^* the conic dual to the image ω of the absolute conic Ω (see section 5.3.2). Therefore we must have

$$\mathcal{P}\mathcal{Q}^*_{\mathcal{B}_0}\mathcal{P}^T \simeq \mathbf{K} \tag{12.37}$$

where $\mathbf{K}$ is the Kruppa matrix of the camera.

It is easy to relate $\mathcal{Q}^*_{\mathcal{B}_0}$ and $\mathcal{Q}^*_{\mathcal{E}}$ using for example proposition 4.24. Using that proposition, $\mathcal{Q}^*_{\mathcal{B}_0} = \mathbf{Q}^{\mathcal{B}_0}_{\mathcal{E}}\mathcal{Q}^*_{\mathcal{E}}\mathbf{Q}^{\mathcal{B}_0\,T}_{\mathcal{E}}$. Using Eq. (12.34), we find the expression of the matrix of the absolute quadric in the projective frame:

$$\mathcal{Q}^*_{\mathcal{B}_0} = \begin{bmatrix} \mathbf{A}\mathbf{A}^T & \mathbf{A}\mathbf{A}^T\mathbf{r}_\infty \\ \mathbf{r}_\infty^T\mathbf{A}\mathbf{A}^T & \mathbf{r}_\infty^T\mathbf{A}\mathbf{A}^T\mathbf{r}_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{K} & \mathbf{K}\mathbf{r}_\infty \\ \mathbf{r}_\infty^T\mathbf{K} & \mathbf{r}_\infty^T\mathbf{K}\mathbf{r}_\infty \end{bmatrix}$$

Plugging back this expression into Eq. (12.37), we obtain exactly the Kruppa constraints of Eq. (12.36).

12.5.6 Dealing with non constant intrinsic parameters

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In practice the assumption that the internal parameters are constant is sometimes somewhat unrealistic. For example, in a sequence of twenty four images of the INRIA library in Sophia-Antipolis of which we show four in figure 12.8, the internal parameters have not been kept constant but it is very reasonable to assume that the skew parameter is equal to 0 and that the ratio $\frac{\alpha_u}{\alpha_v}$ is constant in all views. Is it possible in that case to self-calibrate the views?

The answer to this question is in general yes. Using Eq. (12.37), it should be clear that any constraint on the coefficients of $\mathbf{K}$ can be translated into a constraint on the matrix $\mathcal{Q}^*$. Since this matrix depends upon 8 parameters (a symmetric matrix of rank 3 defined up to scale), one would expect that, given enough views, one could recover its coefficients.

To be more specific, let us consider the zero skew and the constant aspect ratio constraint. As shown in chapter 5, equation (5.37), the zero skew constraint is equivalent to:

$$K_{13}K_{23} - K_{33}K_{12} = 0$$

whereas the ratio $\frac{\alpha_u}{\alpha_v}$ is equal to $\sqrt{\frac{K_{11}-K_{13}^2}{K_{22}-K_{23}^2}}$. For each view i, let us note $\mathbf{K}_i$ the matrix $\mathcal{P}_i\mathcal{Q}^*\mathcal{P}_i^T$. The zero skew constraint leads to the non-linear

Figure 12.8: Four out of twenty four views of the INRIA library in Sophia-Antipolis with varying internal parameters. Courtesy **Sylvain Bougnoux**, INRIA.

minimization of the error function

$$\sum_{i=1}^{n}(K_{i13}K_{i23} - K_{i33}K_{i12})^2$$

with respect to the 8 parameters of $\mathcal{Q}^*$. Once $\mathcal{Q}^*$ has been obtained, one can compute the coefficients of the matrixes $\mathbf{K}_i$ and recover the intrinsic parameters of the cameras. The plane at infinity can be recovered in a variety of manners for example as the null space of $\mathcal{Q}^*$.

Bougnoux [38] proposes to initialize the non-linear minimization in an unconventional way. He first estimates the focal length using the formula in Sec. 12.4.1, then given the intrinsic parameters $\mathbf{A}$, obtains an estimate of $\mathbf{r}_\infty$ by solving a linear least squares system in $\mathbf{A}\mathbf{r}_\infty$ obtained from manipulations of Eq. (12.36).

To continue with the example of figure 12.8, we show in table 12.1 the internal parameters of six of the twenty four views, obtained with that method. It is seen that if α_u, α_v, u_0 and v_0 vary from view to view, the ratio α_u/α_v and the cosine of the skew $\cos(\gamma)$ remain pretty much constant. The focal

α_u	α_v	u_0	v_0	α_u/α_v	$\cos(\gamma)$
616.8	616.8	384	256	1	0
722.1	719.7	361	259	1.003	0.0028
876.0	873.3	319	257	1.003	-0.0032
946.0	947.2	299	250	0.999	-0.0001
1190.5	1193.8	248	255	0.997	0.0010
1019.2	1016.8	318	248	1.002	-0.0017

Table 12.1: This table shows the internal parameters of six of the twenty four views of the INRIA library in Sophia-Antipolis: Courtesy **Sylvain Bougnoux**, INRIA.

lengths, in pixels, of the twenty four views are shown in figure 12.9. Three dimensional Euclidean reconstruction is then possible, as shown in figure 12.10 as well as texture mapping (figure 12.11).

12.6 Computing all the Euclidean parameters

The idea behind all the previous approaches was to use the fact that the rotation matrix is orthogonal in order to eliminate the motion parameters,

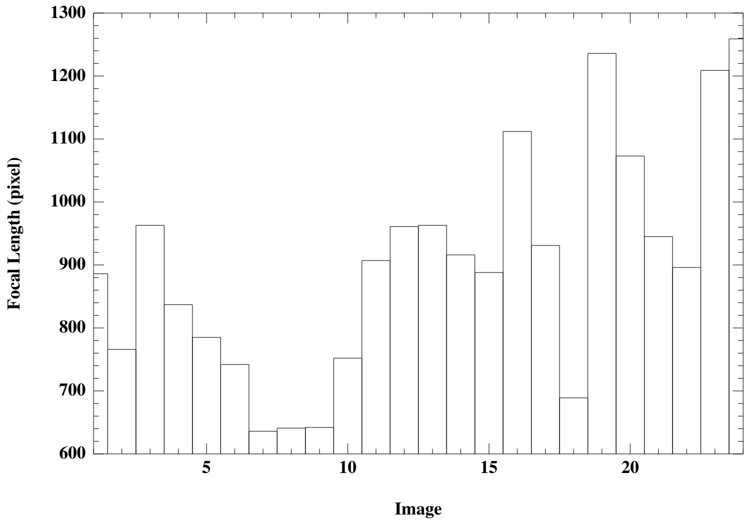


Figure 12.9: The focal lengths (in pixels) of the twenty four views of the INRIA library in Sophia-Antipolis: Courtesy **Sylvain Bougnoux**, INRIA.

relating the projective or affine quantities to Euclidean quantities. By contrast, the methods in this section do so by parameterizing the projective quantities by Euclidean quantities, thus making it possible to compute them directly. This requires that at least the intrinsic parameters and motion parameters are computed simultaneously, so these methods are based on large-scale non-linear optimization and require a good starting point which can be obtained with one of the previous method. Sec. 12.6.1 presents a method based on the global minimization of the epipolar constraints with respect to Euclidean parameters. Sec. 12.6.2 is a global bundle adjustment which involves also the structure. Further results obtained with these methods are shown in Sec. 12.6.3.

12.6.1 Simultaneous computation of motion and intrinsic parameters

The methods previously studied decoupled the estimation of the intrinsic parameters from the estimation of the motion parameters. In effect, the Kruppa equations allow to do this since they *eliminate* the unknown motion parameters. This is nice since, as we saw, the methods involve at most five unknows, and allow semi-analytical, as well as efficient iterative solutions. One disadvantage is that once the intrinsic parameters have been estimated,

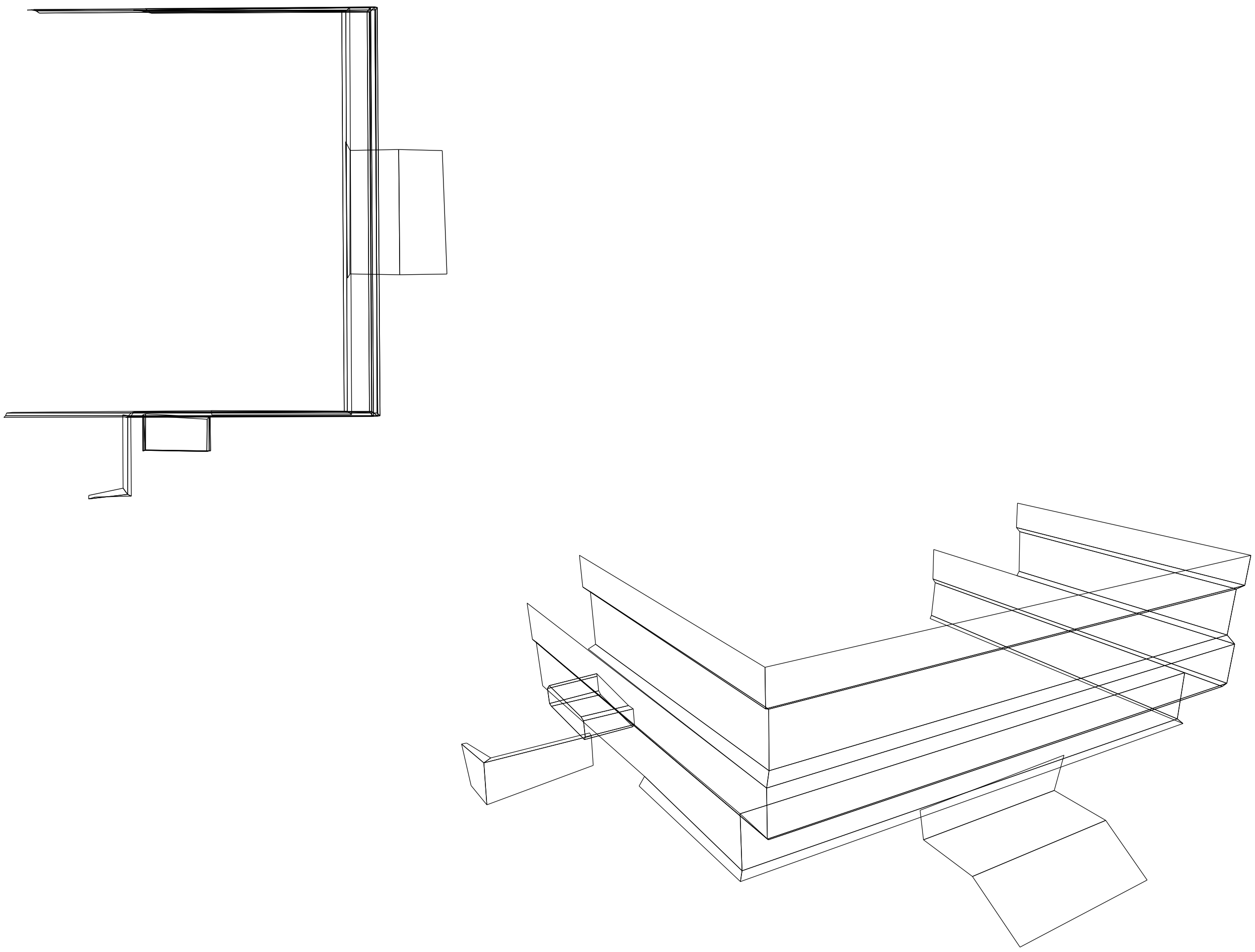


Figure 12.10: Top view and front view of the Euclidean reconstruction of the INRIA library in Sophia-Antipolis. The model has been obtained from a total of twenty four views with varying internal parameters: Courtesy **Sylvain Bougnoux**, INRIA.

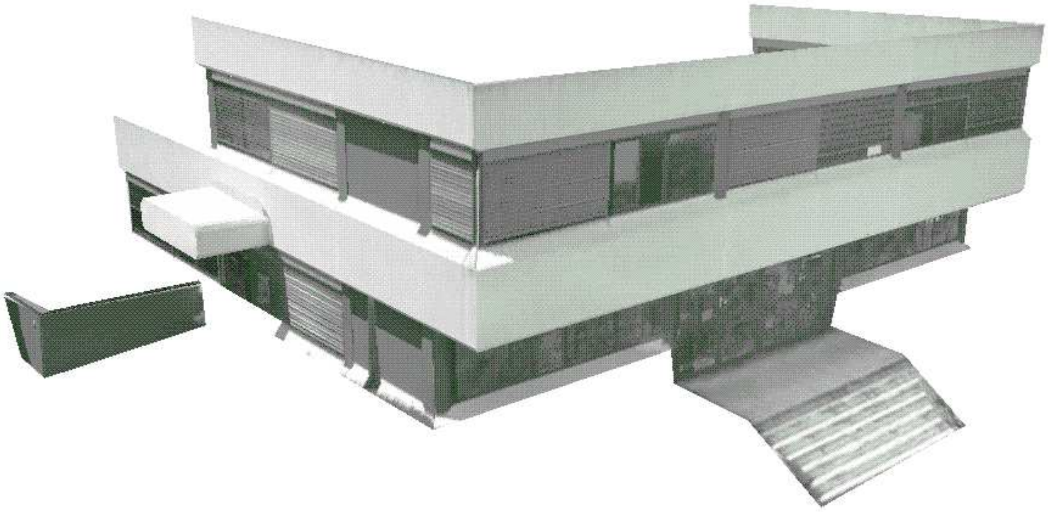


Figure 12.11: Textured front view of the Euclidean reconstruction of the INRIA library in Sophia-Antipolis: Courtesy **Sylvain Bougnoux**, INRIA.

their values cannot be changed, for example as a consequence of a poor estimation of the motion parameters. This problem would show up when trying to decompose the essential matrixes $\mathbf{E}_i$ estimated from $\mathbf{A}^T\mathbf{F}\mathbf{A}$, where $\mathbf{F}_i$ are the estimated fundamental matrices and $\mathbf{A}$ the estimated intrinsic parameters: the constraints (6.34) could possibly be poorly satisfied.

Contrary to this the global method that we describe in this section, estimates simultaneously all the intrinsic and extrinsic parameters. The *form of the parameterization* ensures that the constraints (6.34) on the essential matrixes are satisfied. The basic idea is to consider the error function (7.14) where $\mathbf{F}_i$ is written as an explicit function of the unknowns $\mathbf{A}$, $\mathbf{R}_i$ and $\mathbf{u}_i$ using (6.33) and (6.32). We note (m_{ij}, m'_{ij}) the jth correspondence in the ith pair of views. The error function C is therefore

$$C = \sum_{i=1}^{N} \sum_{j} d^2(\mathbf{m}'_{ij}, \mathbf{A}^{-T}[\mathbf{u}_i]_\times \mathbf{R}_i \mathbf{A}^{-1}\mathbf{m}_{ij}) + d^2(\mathbf{m}_{ij}, \mathbf{A}^{-T}\mathbf{R}_i^T[\mathbf{u}_i]_\times \mathbf{A}^{-1}\mathbf{m}'_{ij}) \tag{12.38}$$

The starting point of the minimization can, for example, be obtained by the methods described in the previous two sections.

The algorithm, which can accomodate N displacements ($N \geq 2$), and, for each displacement i, a minimum of eight correspondences $(\mathbf{m}_{ij}, \mathbf{m}'_{ij})$, can be summarized as follows:

1. Estimate the N fundamental matrices $\mathbf{F}_i$ from the correspondences.

2. Compute an initial estimate of the intrinsic parameters by one of the methods described in sections 12.4.2 and 12.4.3.

3. Compute the N initial motions $(\mathbf{R}_i, \mathbf{t}_i)$ from $\mathbf{F}_i$ and the intrinsic parameters. This is done by computing the Essential matrixes $\mathbf{E}_i = \mathbf{A}^T \mathbf{F}_i \mathbf{A}$. Those matrixes do not usually satisfy the constraints (6.34). They are then corrected by the SVD-based method described in section 6.3.3.

4. Minimize, with respect to the $6N - 1 + int$ variables (we have $3N$ parameters for the N rotations, $3N-1$ parameters for the translations which are globally defined up to scale and int is the number of intrinsic parameters) the error function (12.38).

An interesting feature of this method is that it allows us to take into account some of the constraints between multiple views. For example, in the case of three views, instead of minimizing over the 15 motion parameters $\mathbf{R}_{12}, \mathbf{u}_{12}$, $\mathbf{R}_{23}, \mathbf{u}_{23}$, and $\mathbf{R}_{13}, \mathbf{u}_{13}$, we can replace for example $\mathbf{R}_{13}$ by $\mathbf{R}_{23}\mathbf{R}_{12}$ and $\mathbf{u}_{13}$ by $\alpha \mathbf{u}_{23} + \mathbf{R}_{23}\mathbf{u}_{12}$, where α is the ratio $\|\mathbf{t}_{23}\| / \|\mathbf{t}_{12}\|$. This ratio can be obtained as explained in propositions 11.6, 11.7 and 11.8, or, more efficiently, by observing the images $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ of one point M in the three views.

Let Z_1, Z_2, Z_3 be the depths of M in the three cameras normalized coordinate systems. If $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ are the normalized pixel coordinate vectors, the Euclidean disparity equations can be written (proposition 8.17):

$$\begin{aligned} Z_2 \mathbf{m}_2 &= Z_1 \mathbf{R}_{12} \mathbf{m}_1 + \mathbf{t}_{12} \\ Z_3 \mathbf{m}_3 &= Z_2 \mathbf{R}_{23} \mathbf{m}_2 + \mathbf{t}_{23} \end{aligned}$$

The first equation yields

$$\frac{Z_2}{\|\mathbf{t}_{12}\|} = \frac{\mathbf{u}_{12} \times \mathbf{R}_{12}\mathbf{m}_1}{\mathbf{m}_2 \times \mathbf{R}_{12}\mathbf{m}_1},$$

The second equation yields

$$\frac{Z_2}{\|\mathbf{t}_{23}\|} = \frac{\mathbf{u}_{23} \times \mathbf{m}_3}{\mathbf{m}_3 \times \mathbf{R}_{23}\mathbf{m}_2}$$

The elimination of Z_2 between those two equations yields α.

The number of motion parameters is therefore reduced to 11, and trinocular constraints are enforced by the parameterization. Experiments in [223] have shown that the results are generally better than those of the method based only on the Kruppa equations.

As an example of the kinds of results that can be obtained with this method, we have collected four times five views with a camera. For each set of five views, the intrinsic parameters were kept constant. Each set had a different focal length. To obtain ground truth, a calibration grid was present in one of the images of each of the four sequences. The results are shown in table 12.2. It shows that the best results are obtained for short focal lengths, which yield large fields of view. Although the focal length is overestimated by the method for large values, we notice that the computed aspect ratio is quite consistent over the whole range of focal lengths. We also note that the principal point is not estimated very accurately. However, as shown by the 3D reconstruction errors discussed next, this is not of primary importance. The intrinsic and extrinsic parameters estimated by this method

focal	method	α_u	α_v	u_0	v_0	$\theta - \frac{\pi}{2}$	$\frac{\alpha_u}{\alpha_v}$
9	GRID	481.31	711.54	248.57	260.97	10^{-7}	.6764
	SELFCALIB	503.49	760.71	250.24	282.67		.6618
12	GRID	642.45	950.37	248.30	263.31	-5.10^{-7}	.6759
	SELFCALIB	636.12	921.36	201.52	338.89		.6904
20	GRID	1036.38	1539.6	252.43	272.53	7.10^{-8}	.6731
	SELFCALIB	1208.83	1838.48	251.93	200.58		.6575
30	GRID	1573.20	2330.953	207.98	210.35	4.10^{-7}	.6749
	SELFCALIB	2047.61	3063.94	249.678	198.463		.6682

Table 12.2: Parameters obtained with a zoom camera

have been used to reconstruct points of a known object. Three images have been used and three projection matrices computed from the extrinsic and intrinsic parameters using the formulas of proposition 8.14:

$$\boldsymbol{\mathcal{P}}_1 = [\mathbf{A}\, 0] \qquad \boldsymbol{\mathcal{P}}_2 = \mathbf{A}[\mathbf{R}_{12}\, \mathbf{t}_{12}] \qquad \boldsymbol{\mathcal{P}}_3 = \mathbf{A}[\mathbf{R}_{23}\mathbf{R}_{12}\, \mathbf{R}_{23}\mathbf{t}_{12} + \alpha_2 \mathbf{t}_{23}]$$

We have then performed a 3D trinocular reconstruction from the matched points, The coordinate of the 3D points are obtained in the canonical coordinate system associated of the first camera. They are defined up to scale, since we can reconstruct only up a similarity with the self-calibration technique. In order to compare the reconstruction with the reference data, we have computed the best similarity which relates the two sets of 3D points. After applying this similarity to the initial reconstruction, the final average relative error in 3D space with this sequence is less than 2%.

12.6.2 Global optimization on structure, motion, and calibration

This method also comes as a complement to those described in section 12.4. It uses them to initialize the minimization of an error function that is similar to (7.18).

$$\sum_{i=1}^{n} \sum_{j} d^2(\mathbf{m}_{ij}, \boldsymbol{\mathcal{P}}_i(\mathbf{q}_i)\mathbf{M}_j) \tag{12.39}$$

In this equation, the vectors $\mathbf{q}_i$ represent the parameters used to represent the ith projection matrix $\boldsymbol{\mathcal{P}}_i$. This function is the sum of the squares, ε_{ij}, of the reprojection errors of section 11.3.3.3.

In effect, once relatively good estimates for the intrinsic parameters $\mathbf{A}$, and the extrinsic parameters $\mathbf{R}_i$, $\mathbf{t}_i$ have been obtained, the minimization of the function (12.39) with respect to those parameters and the coordinates of the m points M_j can be carried out to refine the results. This is similar in spirit to the method described Sec. 12.6.1. Compared with the minimization of the set of epipolar constraints described in that section, the minimization of the reprojection error has the advantage of providing two constraints per point, and can therefore be expected to yield better results when many images are available. However, it requires the computation of the 3-D coordinates of each point, and therefore is computationally more expensive. It has been found [136, 425] that this stage modifies only slightly the intrinsic and extrinsic parameters obtained by the Kruppa or the canonic transformation method.

Fig. 12.6.2 shows an example of reconstruction obtained with this method, starting from a sequence of fifteen images. Four of them are shown in Figure 12.6.2. Since a 3D model of the scene is not available, to assess the precision of the results, a few angles and ratio of lengths are computed in Fig. 12.6.2.

12.6.3 More applications

We now show some more results obtained from the self-calibration methods that have been described.

Figure 12.15 and show an application to the construction of a 3D model of a complicated site. The models are obtained in several stages. First, images of the site are acquired from as many viewpoints as possible in order

Figure 12.12: Some views of a sequence of 15 images of the Valbonne church

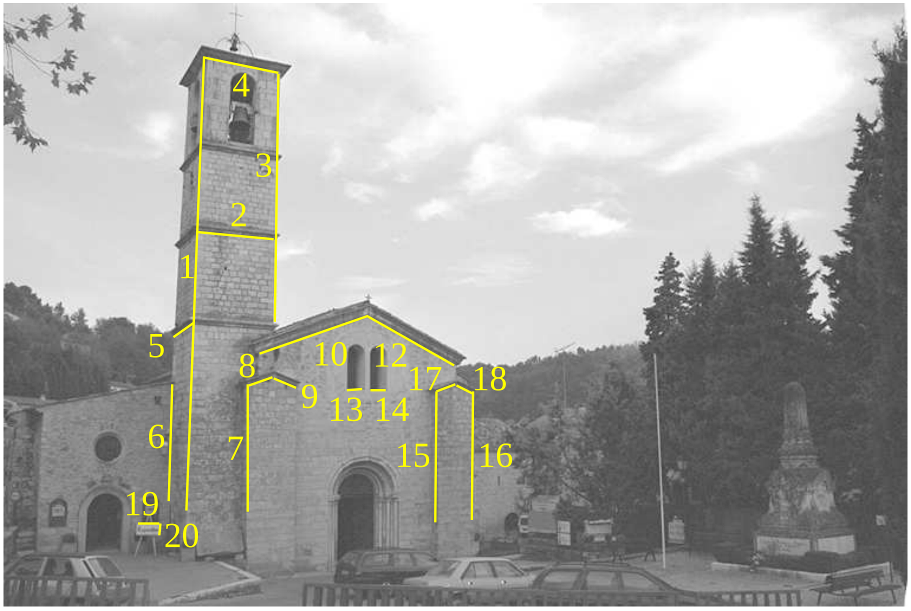
to guarantee full coverage. Next, the views are calibrated using one of the techniques that have been described in this chapter. This is an automatic process. Third and last, the views and the calibration parameters are used in a modelling system to construct a 3D Euclidean representation of the scene. This is a partially manual operation. The model shown in figure 12.15 and have been obtained from fourty five images with constant intrinsic parameters. These parameters have been obtained by combining the methods described in sections 12.4.3, 12.6.1 and 12.6.2. Note that, contrary to photogrammetric methods, no reference points with known coordinates have been used.

The images that have been used to build this three-dimensional model can be used to map textures on the polyhedral description using some standard computer graphics techniques. One result is shown in figure 12.16.

A similar process has been applied to a set of images of the Arcades square (some of them shown in figure 11.8. Two views of the resulting geometric model is shown in figure 12.17. The rendering effect obtained by using the images to texture the polyhedral model is shown in figure 12.18. Figure 12.19 shows a different viewpoint as well as some of the estimated cameras positions and orientations.

The next example demonstrate the capacity of this class of methods to deal with continuous streams of images for applications to augmented reality. Given a number of images of a single scene, we add to the scene one or more artificial objects while respecting the geometry of the scene and/or the movements of the objects in the scene. A first example was shown in Figs.

Figure 12.13: Some line segments, and the corresponding 3D reconstruction



Angle	computed	real	Ratio	computed	real
$\cos(1,3)$	1	1	$2/4$	0.993	1
$\cos(1,2)$	0.054	0	$4/5$	0.999	1
$\cos(3,5)$	0.032	0	$10/12$	1.05	1
$\cos(6,7)$	0.999	1	$8/9$	1.004	1
$\cos(15,16)$	1	1	$17/18$	0.966	1
$\cos(19,20)$	0.101	0	$13/14$	0.902	1

Figure 12.14: Some line segments, and the corresponding measurements

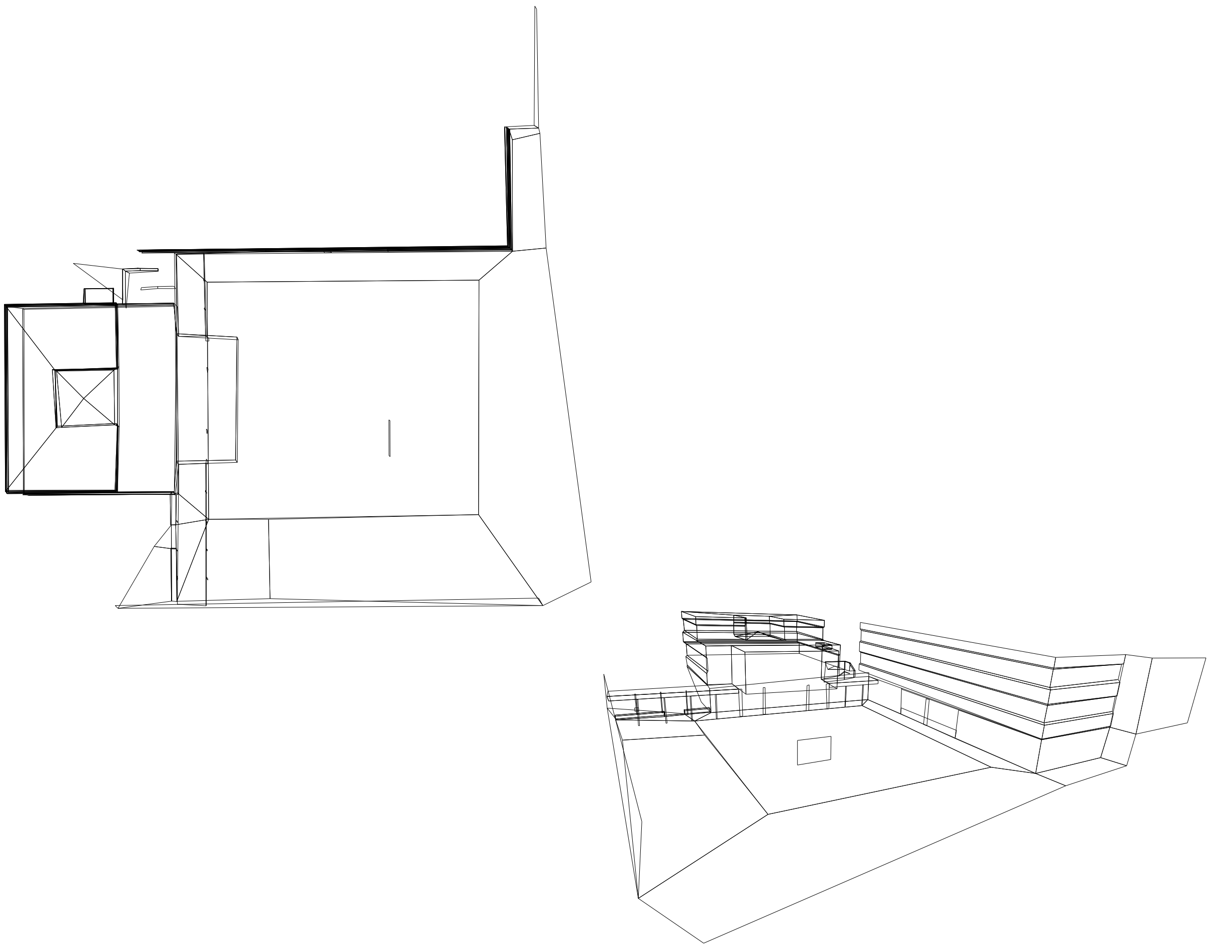


Figure 12.15: Top and front view of the Euclidean reconstruction of the INRIA site in Sophia-Antipolis. The model has been obtained from a total of fourty five views: Courtesy **Sylvain Bougnoux**, INRIA.

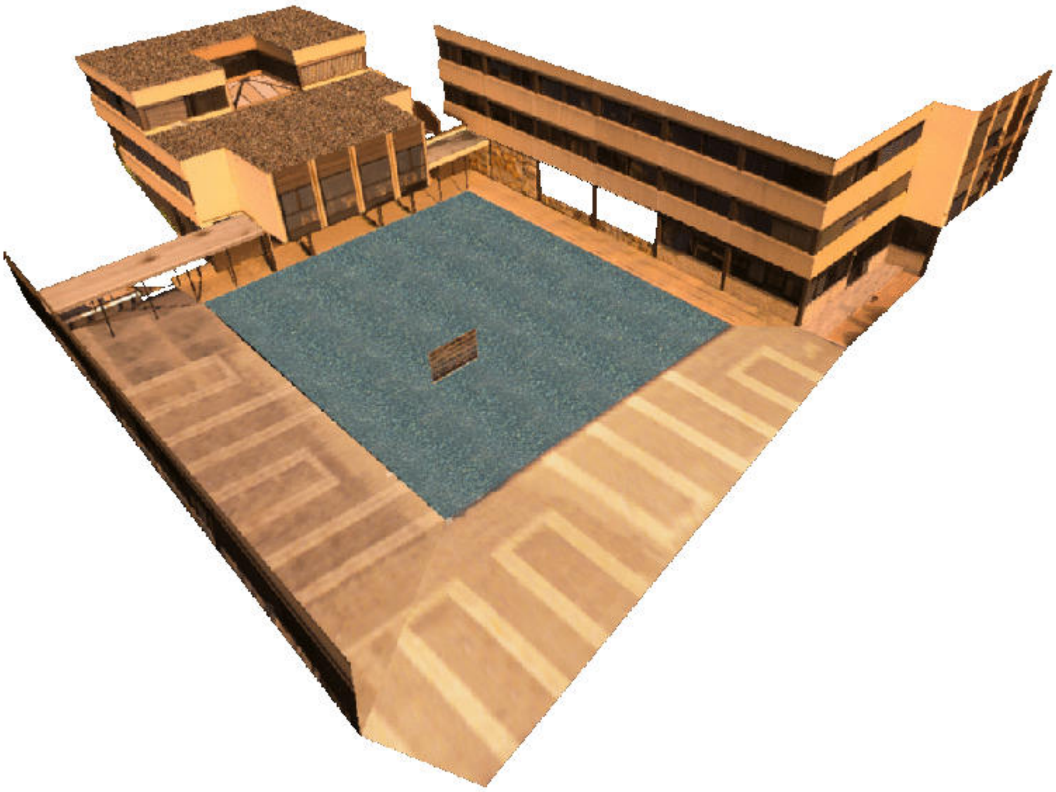


Figure 12.16: Textured front view of the Euclidean reconstruction of the INRIA site in Sophia-Antipolis: Courtesy **Sylvain Bougnoux**, INRIA.

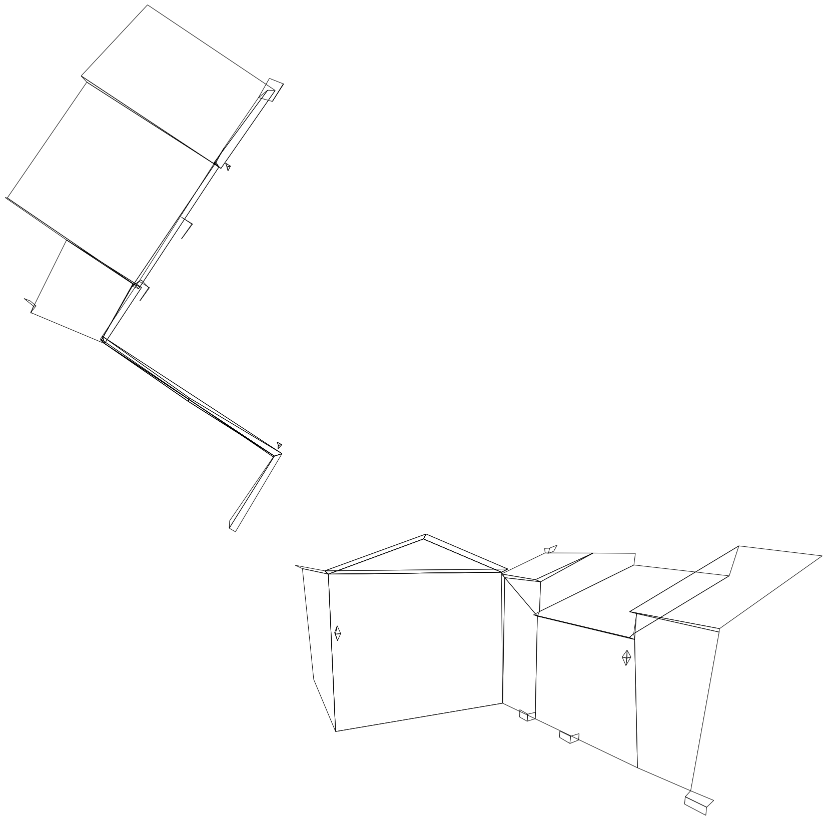


Figure 12.17: Top and front view of the Euclidean reconstruction of the Arcades square in Valbonne. The model has been obtained from a total of ? views: Courtesy **Sylvain Bougnoux**, INRIA.

Figure 12.18: Textured view of the Euclidean reconstruction of the Arcades square in Valbonne: Courtesy **Sylvain Bougnoux**, INRIA.

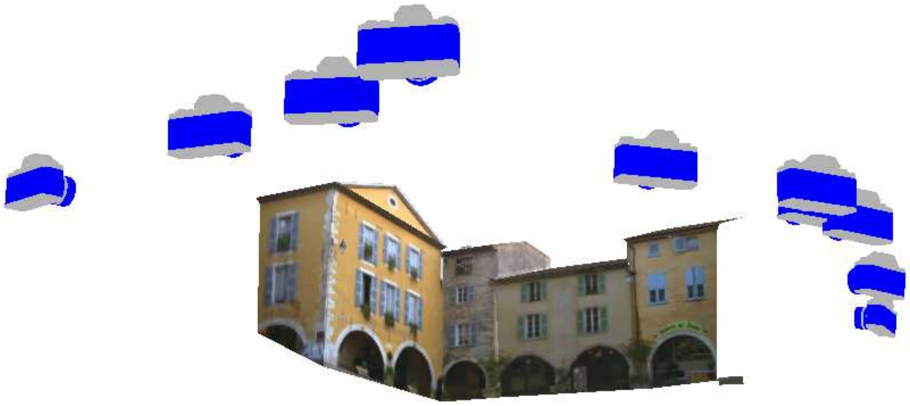


Figure 12.19: Another textured view of the Euclidean reconstruction of the Arcades square in Valbonne showing the estimated positions and orientations of some of the cameras: Courtesy **Sylvain Bougnoux**, INRIA.

2.4, 2.5, 2.6. The second sequence is a video sequence taken with a handheld camera (figure 12.20 shows three images extracted from that sequence). The scene is static and the only movement is that of the camera. The intrinsic parameters are changing over the sequence.

The sequences are then processed as follows. To begin with, characteristic points are tracked automatically in the sequence with a technique based on image correlation. We then manually select a number of views of the sequence corresponding to sufficiently different positions of the camera so that the estimation of the fundamental matrices between the pairs of views is robust. For the second sequence we first identify subsequences where the intrinsic parameters are roughly constant and then select the views. The next step is to estimate the fundamental matrices and to solve the system of the Kruppa equations by the methods described in sections 12.4.3, 12.6.1, so as to obtain an initial estimate of the intrinsic parameters of the camera. Thus we have access to an Euclidean reconstruction of the scene and consequently to the position and the relative orientation of the cameras in the space at the chosen instants. Those initial estimates are then refined by the technique described in section 12.6.2.

After this step, we now have a more precise estimate of the intrinsic parameters and the relative positions and orientations of the cameras at chosen times. Then we can obtain this position at any time of the sequence by simple interpolation of the intrinsic (for the second sequence) and extrinsic parameters. The interpolated quantities are the intrinsic parameters, the

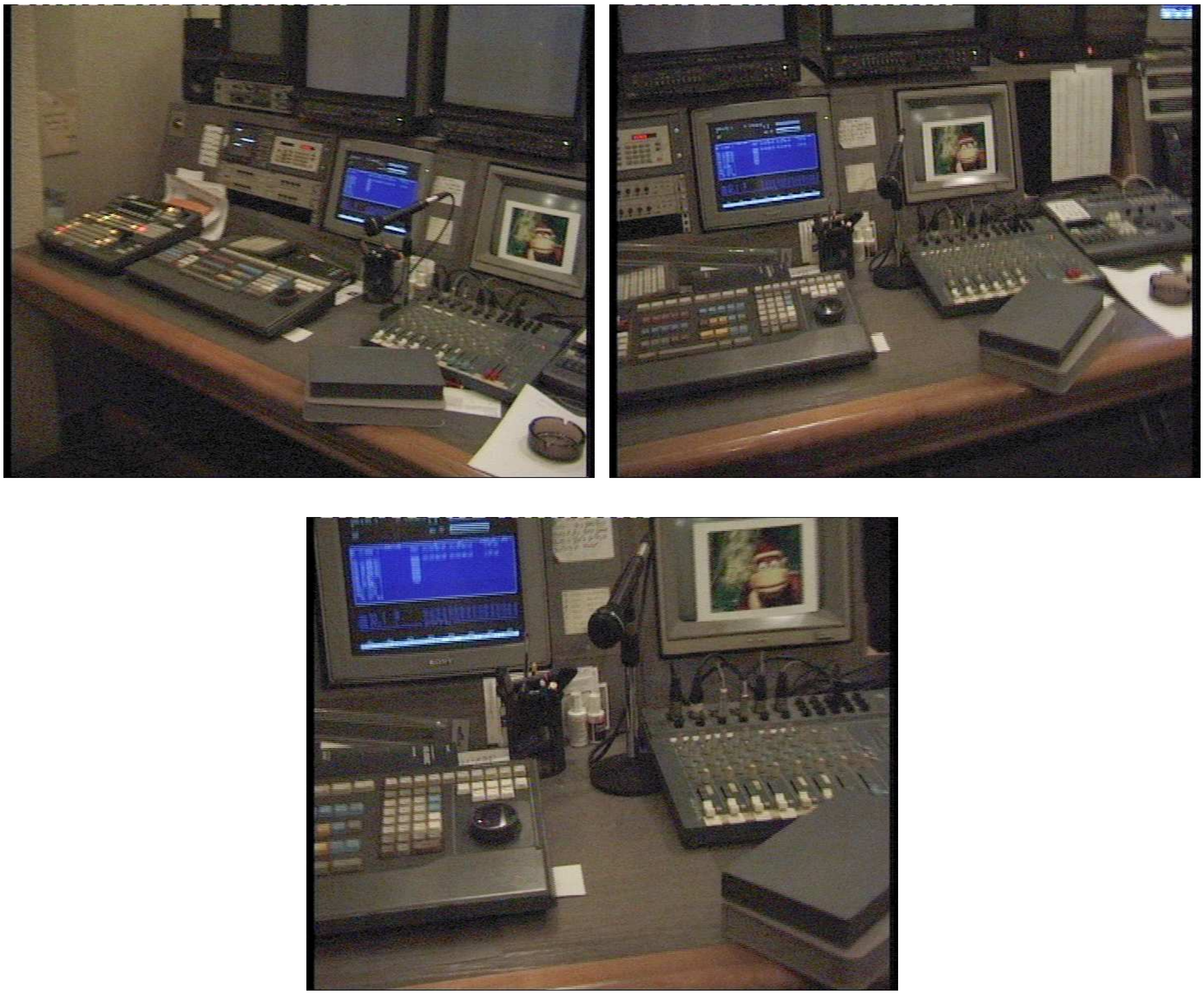


Figure 12.20: Three images of the sequence taken with a handycam: Courtesy **Luc Robert**, INRIA.

translation and the rotation. Only the rotation is a problem since the structure of the rotation group is a bit complicated, but using a parametrization with Euler angles or quaternions leads to a satisfactory solution.

The relative positions of the camera being known after the above computations, we can easily model by stereo the part of the scene we want to modify, a planar scene for the helicopter sequence, part of the table with a box sitting on top for the handycam sequence. This three-dimensional model is then transfered into a CAD system which will "put on top" another virtual object like the power plant in figure 2.5 or the lamp in figure 12.21. A "reasonable" source of light can be positioned with respect to the set "ground+power plant" for the first sequence or "table+box+lamp" in order to compute shadows with the help of classical image synthesis techniques. The model is then reinserted into the initial sequence by combining the initial intensities of the ground, in the first example, of the table and the box in the second, with the shadows.

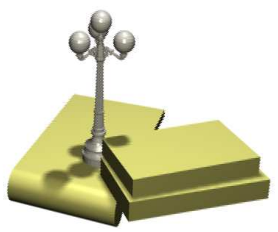


Figure 12.21: Model to insert in the sequence of figure 12.20.

We show in figures 2.6 and 12.22 three images of the modified sequences. These images correspond to those of figures 2.4 and 12.20. The results are quite remarkable: the synthetic objects appear, as they should, to be static with respect to the environment. The impression is even stronger when looking at the modified sequences.

Figure 12.22: Results obtained from images 12.20 and 12.21.

12.7 Degeneracies in self-calibration

In this section we sketch out an analysis of the problem of degeneracies in self-calibration. We start with a description of the *intrinsic* degeneracies, i.e. of those which are present *independently* of the algorithm which is used for solving the calibration problem, e.g. solving the Kruppa equations. We then describe some particular degeneracies that are due to the particular type of algorithm used.

Let us therefore restate the calibration problem (i.e. the upgrade of the representation from projective to Euclidean) as that of recovering a plane in projective space and a conic Ω in that plane which is proper (i.e. non degenerate) and imaginary and such that all its images are also proper (they are also imaginary since the projection matrixes are real).

We explore the possibility that there may be several solutions to this problem. Any such solution will be called a *spurious* solution to the calibration problem. Intuitively, the reason why there may exist such spurious solutions is that we usually perform a finite number of displacements with the camera and only enforce that the images of the absolute conic are identical for those views, whereas the absolute conic satisfies this property for all possible displacements. A particular sequence of displacements for which there exist spurious absolute conic will be called a *critical sequence.*

Our exposition follows that of Sturm [349]. We consider only the simpler case where the spurious conics lie in the true plane at infinity. The case where they lie in a plane that is not the real plane at infinity is discussed in [349]. We will nonetheless obtain a fairly complete description of the intrinsic critical sequences, i.e. of the sequences that are critical for self-calibration independently of the particular algorithm used for self-calibration. We will then describe some critical sequences that are specific to a given algorithm.

12.7.1 The spurious absolute conics lie in the real plane at infinity

Let us assume that we have n views and choose the world coordinate system as the normalized coordinate system of one of the cameras, e.g. the first, and the image coordinate systems as the intrinsic coordinate systems. All projection matrixes are therefore of the form

$$\boldsymbol{\mathcal{P}}_i \simeq [\mathbf{R}_i \, \mathbf{t}_i]$$

where the matrixes $\mathbf{R}_i$ are rotation matrixes.

A proper imaginary conic Ω in Π_∞ is represented by a definite symmetric 3×3 matrix $\mathbf{C}$ and the matrix $\mathbf{c}_i$ of the image ω_i of Ω in view i is

$$\mathbf{c}_i \simeq \mathbf{R}_i \mathbf{C} \mathbf{R}_i^T$$

The images of Ω in two views will be identical if and only if their matrices are proportional ($\mathbf{c}_i \simeq \mathbf{c}_j$) and, since the $\mathbf{R}_i$'s are rotation matrixes (therefore of determinants equal to 1), this is equivalent to

$$\mathbf{R}_i \mathbf{C} \mathbf{R}_i^T = \mathbf{R}_j \mathbf{C} \mathbf{R}_j^T \; i \neq j, \tag{12.40}$$

note the $=$ sign. The analysis of this equation yields a complete characterization of the critical sequences of displacements and of the spurious absolute conics.

Proposition 12.12 (Sturm) *The critical sequences of displacements with respect to a proper imaginary conic in the plane at infinity are those for which the relative rotations between any two views of the sequence fall into one of three cases:*

1. *A rotation of angle 0 (pure translation case).*
2. *An arbitrary rotation around a fixed axis* $\mathbf{r}$.
3. *A 180 degrees rotation around an axis perpendicular to a fixed direction* $\mathbf{r}$.

In the case 1, the cameras are translated with respect to each other and all proper imaginary conics in the plane at infinity are spurious absolute conics. In the cases 2 and 3, the spurious absolute conics form part of a one parameter linear family of conics defined by the matrixes:

$$\mathbf{C} = \lambda \mathbf{I}_3 + \mu \mathbf{r}\mathbf{r}^T$$

where λ *and* μ *are two arbitrary parameters.*

Proof : Rewrite equation (12.40) as

$$\mathbf{R}\mathbf{C} = \mathbf{C}\mathbf{R} \tag{12.41}$$

where $\mathbf{R} = \mathbf{R}_j^T \mathbf{R}_i$ is the relative rotation between the two views. This equation shows that the matrix $\mathbf{R}$ must leave the eigenspaces of the symmetric definite matrix $\mathbf{C}$ invariant. There are only three possible cases corresponding to the number of distinct eigenvalues.

Three equal eigenvalues Ω is the real absolute conic. the eigenspace of $\mathbf{C}$ is the whole space and therefore $\mathbf{R}$ must be the identity matrix.

Two equal eigenvalues The corresponding eigenspace is a plane Π. The eigenspace corresponding to the simple eigenvalue is of dimension 1, it is defined by a vector $\mathbf{r}$ orthogonal to the plane. The rotations which leave Π and $span(\mathbf{r})$ globally invariant are the identity, all the rotations of axis $\mathbf{r}$ and the rotations of 180 degrees around an axis in Π, i.e. orthogonal to $\mathbf{r}$. They form a group.

Three different eigenvalues The three eigenspaces are orthogonal and of dimension 1. The rotations which leave those three eigenspaces globally invariant are the identity and all the 180 degrees rotations around any of the three directions defined by the three eigenvectors. They form a subgroup of the previous one.

Conversely, if the relative rotation $\mathbf{R}$ is one of the above, let us show that there exist spurious absolute conics. If $\mathbf{R}$ is the identity then equation (12.41) is true for all definite symmetric matrixes $\mathbf{C}$: all proper imaginary conics in the plane at infinity are spurious absolute conics in the case of camera translation. Otherwise, let $\mathbf{r}$ be the vector defining the axis of rotation, and define the family of symmetric matrixes

$$\mathbf{C} = \lambda \mathbf{I}_3 + \mu \mathbf{r}\mathbf{r}^T \tag{12.42}$$

where λ and μ are arbitrary real numbers. Any rotation $\mathbf{R}$ around $\mathbf{r}$ satisfies the relations $\mathbf{R}\mathbf{r} = \mathbf{r}$ and $\mathbf{R}^T\mathbf{r} = \mathbf{r}$ and therefore (12.41) is satisfied. The matrix $\mathbf{C}$ is positive definite if and only if its eigenvalues are positive. The eigenvalue corresponding to the eigenspace of dimension 1 generated by $\mathbf{r}$ is readily seen to be equal to $\lambda + \mu\|\mathbf{r}\|^2$ and the double eigenvalue corresponding to the plane Π orthogonal to $\mathbf{r}$ is equal to λ. Therefore there are two infinite subfamilies of definite matrixes of the linear family defined by (12.42). The first one is defined by the relations $\lambda > 0$ and $\lambda + \mu\|\mathbf{r}\|^2 > 0$, the second one by the relations $\lambda < 0$ and $\lambda + \mu\|\mathbf{r}\|^2 < 0$. $\square$

The practical implications of this proposition are important since some of the critical sequences are often used for acquiring sequences of images. An example is the case where the camera is translating arbitrarily while rotating by an arbitrary angle around a fixed direction (pure translation is a special case that has been identified in the previous section as a degenerate case of the Kruppa equations), see figure 12.23. Another special case is the case of the planar motion, often encountered in mobile robotics, where the translations are limited to a fixed plane and the rotations are around the direction

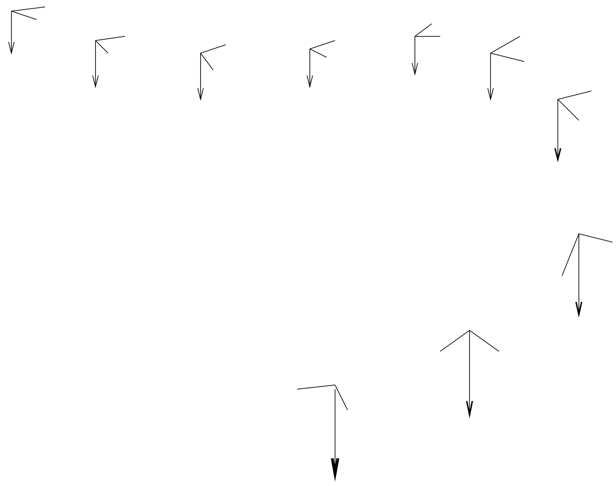


Figure 12.23: The motion of a camera translating arbitrarily while rotating around a fixed direction is critical with respect to self-calibration.

orthogonal to the plane, see figure 12.24. Yet another special case is that

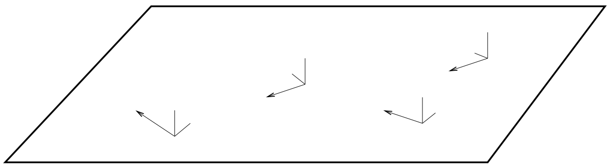


Figure 12.24: The motion of a camera translating in plane and rotating around an axis perpendicular to the plane is critical with respect to self-calibration.

of the orbital motion where the motion is planar and the camera is rotating around a point of the plane and pointing to that point. This kind of motion is often used when acquiring images of an object for 3D modelling, see figure 12.25.

12.7.2 Degeneracies of the Kruppa equations

In the last two sections, we have described intrinsic degeneracies of the self-calibration process. One might expect that a particular algorithm would suffer from those degeneracies and, perhaps, from others. This is the case of the class of algorithms that solve the Kruppa equations: there exist non-

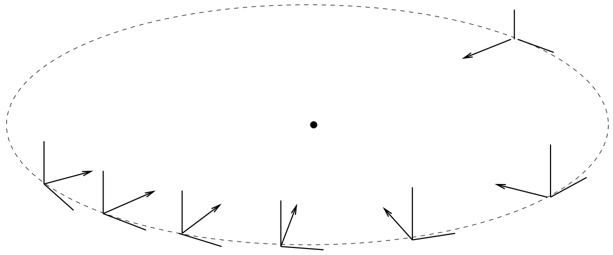


Figure 12.25: The motion of a camera rotating in plane around a fixed point and pointing to that point is critical with respect to self-calibration.

intrinsic critical sequences and spurious absolute conics. We describe such a degeneracy in the next proposition.

Proposition 12.13 *When the optical centers lie on a sphere and the optical axes go through the center of the sphere there exists a one-parameter family of spurious solutions to the Kruppa equations. The ratio* $\frac{\alpha_u}{\alpha_v}$ *cannot be recovered from the sequence of views.*

Proof : Consider a sphere with a smaller radius than the one of the sphere on which the optical centers lie. The cones tangent to that sphere whose vertexes are the optical centers intersect the retinal planes along identical circles centered at the principal points. In the pixel coordinate frames, those circles appear as ellipses. Consider now a pair of cameras: there are two epipolar planes that are tangent to the sphere and the corresponding epipolar lines are tangent to the two ellipses in the image planes. Therefore those ellipses satisfy the Kruppa equations.

Note that since the ellipses are real, they cannot be the images of a spurious absolute conic. But imagine that we increase the radius of the sphere so that the optical centers are inside the sphere. The tangent cones are now imaginary and intersect the image planes along imaginary circles which are still centered at the principal points. In the pixel coordinate frames those imaginary circles are proper imaginary ellipses. Similarly to the real case, if we consider a pair of optical centers, the corresponding pair of proper imaginary ellipses satisfy the Kruppa equations.

When we vary the radius of the sphere these ellipses only change by a scale factor: all the internal parameters remain the same, except α_u and α_v that are scaled by the same factor. In particular, their ratio remains constant. □

Table 12.3: Summary of the variables involved in the minimization based approaches for self-calibration with m views and n points

Method	Initialization		Projective stage		Euclidean stage	
Kruppa	u_0, v_0	(2)	$\mathbf{F}_i$	$(m-1)\times(7)$	$\mathbf{K}$	(5)
Epipolar/Motion	$\mathbf{A}$	(4)	$\mathbf{F}_i$	$(m-1)\times(7)$	$\mathbf{A}$, $\mathbf{R}_i, \mathbf{t}_i, \alpha_i$	$(5+6m-1)$
Canonic	$\mathbf{r}_\infty$	(3)	$\boldsymbol{\mathcal{P}}_i$, $\mathbf{M}_j$	$(11(m-1)+3n)$	$\mathbf{r}_\infty, \mathbf{A}$	$(5+3)$

This degeneracy of the Kruppa equations is interesting in practice and in theory. It explains why the methods which start from the projection matrices (Sec. 12.5) have generally been found to be more stable than the methods which start from the Fundamental matrices (12.4).

There does not exist a spurious absolute conic in this case. But there exists a spurious absolute *quadric*, i.e. the sphere. The reason for this is that the Kruppa equations do not enforce the planarity of the absolute conic.

12.8 Discussion and conclusion

The approach based on the polynomial system of equations have the advantage of not requiring an initialization. However, they have several practical drawbacks, which are mainly the complexity and the difficulty to use multiple views.

The minimization based approaches are designated according to the minimization equation used to obtain the Euclidean parameters. They require a starting point, which can be obtained with the help of physical considerations on the intrinsic parameters ("Kruppa", and the "Epipolar/Motion" method), or the Cheirality inequalities (the "Canonic" method). The Table 12.8 summarizes the variables which are involved in the minimization based approches that were considered: All the three methods require as a first stage the computation of the Fundamental matrices, which encode the basic projective information. Note that this information is contained in the projection matrices computed in the third method. The remaining information can be thought of as (a) the canonic parameters allowing to convert the representation to Euclidean. (b) the "additional motion parameters" which ensure the consistency of the set of views, (c) the structure parameters. The Kruppa method allows to compute sequentially the intrinsic parameters $\mathbf{A}$, and then the motion parameters $(\mathbf{R}_i, \mathbf{t}_i)$, followed by the scale factors α_i, and then the structure parameters. The Epipolar method recomputes, in the second stage, some projective quantities, as well as the "additional motion parameters", and the canonic parameters. The Canonic method computes the "additional motion parameters" in the first stage, as well as the structure parameters. The second stage involves only the canonic parameters. It seems that there is a trade-off between computational efficiency, and robustness to noise with multiple views. With a large number of views, the methods seem to be more and more robust in the order where they are listed. With a small number of views, the latter method can have more convergence problems.

All of the methods can be refined by a global optimization over all the parameters of the problem, as described in Sec. 12.6.2, which is even computationally more expensive. This method does not have a projective stage, and rather requires a very good initialization.

Experiments in [223, 438] show that the motion and the 3D reconstruction are not very affected by large variations of the position of the principal point. In fact, [63] show that the most important parameter is the aspect ratio, which turns out to be the parameter which is the most precisely estimated. The very reason why some parameters are not precisely estimated by the self-calibration methods is that their influence on the 3D reconstruction is small.

12.9 References and further reading

General methods The Kruppa equations have been discovered by Faugeras and Maybank [110] and named after the Austrian mathematician Erwin Kruppa who had been working on problems of photogrammetry [199]. They were used to develop a the first theory of self-calibration [242]. The first working general self-calibration algorithms are based on those equations [99, 228, 223]. Zeller [425, 426] proposes a method to initialize them, and to solve them accurately using the uncertainty of the Fundamental matrix. A detailed algebraic study of those equations is also done. The simplification of the Kruppa equation based on an SVD of the Fundamental matrix was first used in [228], then reformulated in [140] and more simply in [221], where it is the basis for a self-calibration algorithm. In the same spirit, Mendonçca and Cipolla [247] minimize the residual of the singular values constraints (Huang-Faugeras) on the essential matrix, which is expressed as a function of the fundamental matrices and the unknown intrinsic parameters. All those methods are based on reprojected constraints, and use the fundamental matrix.

The first self-calibration method which operates in $\mathbb{P}^3$ and starts from the projective reconstruction was due to Hartley [134, 142]. The methods rely heavily on an intermediate localization of the plane at infinity which was eventually done by partitioning [133]. Latter, Heyden and Astrom [158] introduced an easier way to express Euclidean constraints in the projective reconstruction, and Triggs [378] gave it a concise formulation in terms of the absolute quadric, which made it a popular method, able to tackle naturally the varying intrinsic parameters. Some numerical refinements were presented by Oliensis [263].

Self-calibration with varying intrinsic parameters was first theoretically investigated by Heyden and Astrom [159], who showed that four zero-skew and unit aspect-ratio cameras yield a Euclidean reconstruction. They subsequently got the very general theoretical results that it is sufficient to that just one of the intrinsic parameters is known [161], or constant [162]. Pollefeys [271] addressed at the same time the less general case of varying focal length, and then [273] showed that it is sufficient to know the skew. Before the theoretical proofs, practical solutions were proposed. Encisso et al [93] relied on minimization of calibration and motion parameters. Bougnoux [38] expresses the constraints of unit aspect-ratio and zero skew on the Euclidean projection matrices in order to constrain the canonic transformation parameters.

Particular cases Self-calibration for an affine camera was proposed by Quan [280]. Kahl and Heyden [164] first use an affine camera, that they self-calibrate. The resulting Euclidean parameters initialize a perspective bundle adjustment.

Several authors [135, 275, 300] have studied the problem of extracting the focal lengths from the fundamental matrix, assuming that the other intrinsic parameters are known. Brooks et al [45] point out to degenerate cases which occur for a verging stereo system with coplanar optical axes. Of course, this is a particular case of a situation we described in this chapter. They find that self-calibration of the focal lengths is possible only if the focal lengths are assumed to be identical, and provided that the inwards vergences are not identical. In [81], they perform an error analysis in the neighboorhood of degenerate configurations and show that this error can be considerable. Using the Kruppa equations, Bougnoux [38] is able to obtain a closed-form solution which makes it possible for him to study in [39] the degenerate configurations.

The possibility of recovering constant intrinsic parameters from the infinity homography was pointed to by Luong and Vieville [227] and studied in more detail by Hartley [142]. The method was extended to variable intrinsic parameters by Seo and Hong. In [313], they prove that self-calibration of a stationary camera with varying intrinsic parameters and zero skew is unique, and they study the effect of the error on the principal point on the recovered focal. Agapito et al [80] propose a simpler algorithm by remarking that solving for the image of the absolute conic instead of its dual makes it possible to express linearly certain constraints (such as zero-skew).

Vieville [396] gives an algorithm for self-calibration of a camera undergoing a axis rotation. Armstrong et al [7] reformulate the problem in terms of estimating fixed points and propose a solution using the trifocal tensor. In a similar vein, Faugeras et al [103] describe self-calibration of a 1D camera based on the trifocal tensor of three 1D images. They prove that a 2D camera undergoing a planar motion reduces to a 1D camera.

Planar scenes are typically degenerate cases, since the Fundamental matrix, as well as the projective reconstruction, cannot be estimated uniquely. Triggs [380] showed that five views of a plane is sufficient to recover all the intrinsics parameters of a camera, less if some parameters are fixed a priori. The basic constraint is that the projections of the two circular points of the 3D plane must lie on the image of the absolute conic.

Although a few critical cases were pointed to before, the first systematic classification of critical sequences for self-calibration is due to Sturm [349]. This was latter extended to variable focal lengths. Assuming that focals are the only unknowns, Bougnoux [39] obtains a classification and parameterization. In [38] he gave an explanation for the unstability of the recovered intrinsic parameters. Other work including more general hypothesis such as zero-skew, known aspect ratios is [351, 184]. In the case of constant intrinsic parameters, Ma et al [231] characterize explicitly the ambiguities in calibration and recontruction in the critical cases, in terms of subgroups, and apply the analysis to derive the ambiguity in the synthesis of new views. Zisserman et al [444] study in great detail the one-parameter ambiguity which arises when the camera rotations are about axes with a common direction, and in particular when does an additional constraint not resolve it.

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