Lecture 7
Displacement from points and lines

Olivier Faugeras

Definition of the E-matrix

- We assume a pair of *calibrated* cameras.
- Choose the world coordinate system $\mathcal{F}_O$ equal to the canonical coordinate system of the first camera $\mathcal{F}_C$ and write:

$$\mathbf{Q}_{\mathcal{F}_C}^{\mathcal{F}_{C'}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- $\mathbf{R}$ is the 3×3 rotation matrix between $\mathcal{F}_C$ and $\mathcal{F}_{C'}$ and $\mathbf{t}$ is the vector $\overrightarrow{\mathbf{C}'\mathbf{C}}$ expressed in $\mathcal{F}_{C'}$.
- Write that the three vectors $\mathbf{t}, \overrightarrow{\mathbf{Cm}}_{\backslash \mathcal{F}_{C'}}, \overrightarrow{\mathbf{C}'\mathbf{m}'}$ are coplanar:

$$\mid \mathbf{t},\, \mathbf{Rm},\, \mathbf{m}' \mid = \mathbf{m}'^T [\mathbf{t}]_\times \mathbf{Rm} = 0$$

- The matrix

$$\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$$

is called the Essential matrix.
- The equation is called the Longuet-Higgins equation.

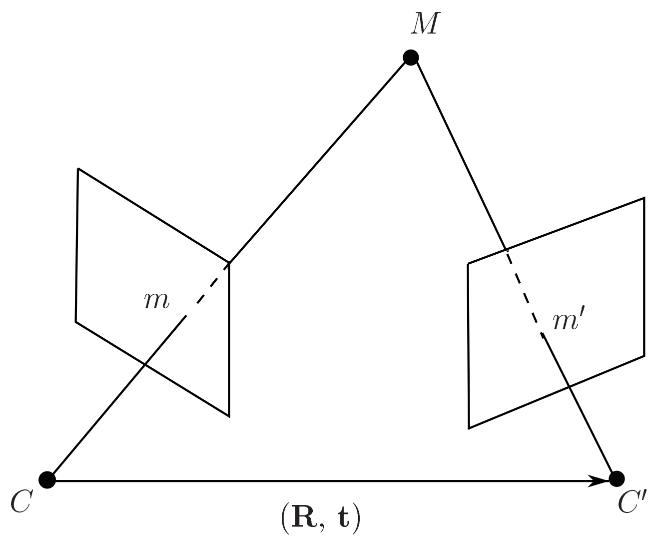

Figure 1: Constraint on the motion parameters derived from point matches.

A characterization based upon linear algebra

Theorem 1 *A real* 3×3 *matrix* $\mathbf{E}$ *can be decomposed as the product* $[\mathbf{t}]_\times \mathbf{R}$ *of an antisymmetric matrix* $[\mathbf{t}]_\times$ *and a rotation matrix* $\mathbf{R}$ *if and only if it has one singular value equal to 0 and the other two singular values are equal.*

- The proof uses the following two important lemmas.

Lemma 1 *The essential matrix* $\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$ *satisfies* $\mathbf{E}^T \mathbf{t} = 0$.

Lemma 2 *The essential matrix* $\mathbf{E} = [\mathbf{t}]_\times \mathbf{R}$ *is such that* $\mathbf{E}\mathbf{E}^T$ *depends only on the translation vector* $\mathbf{t}$.

A characterization based upon linear algebra (*continued*)

- Write:

$$\det(\mathbf{E}\mathbf{E}^T - \lambda \mathbf{I}) = -\lambda^3 + a\lambda^2 + b\lambda + c$$

- Since $c = \det(\mathbf{E}\mathbf{E}^T) = \det^2(\mathbf{E})$, we have $c = 0$.
- The condition on the equality of the singular values is equivalent to:

$$a^2 + 4b = 0$$

A characterization based upon linear algebra (*continued*)

- Express this expression as a function of the elements of $\mathbf{E}\mathbf{E}^T$ and obtain the following two propositions:

Proposition 1 *A real* 3×3 *matrix* $\mathbf{E}$ *with row vectors* $\mathbf{e}_i^T$ *can be decomposed as the product* $[\mathbf{t}]_\times \mathbf{R}$ *of an antisymmetric matrix* $[\mathbf{t}]_\times$ *and a rotation matrix* $\mathbf{R}$ *if and only if*

$$\mathbf{e}_1^T(\mathbf{e}_2 \wedge \mathbf{e}_3) = 0$$

$$(\|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \|\mathbf{e}_3\|^2)^2 = 4(\|\mathbf{e}_1 \wedge \mathbf{e}_2\|^2 + \|\mathbf{e}_2 \wedge \mathbf{e}_3\|^2 + \|\mathbf{e}_3 \wedge \mathbf{e}_1\|^2)$$

Proposition 2 (Huang-Faugeras) *A real* 3×3 *matrix* $\mathbf{E}$ *with row vectors* $\mathbf{e}_i^T$ *can be decomposed as the product* $[\mathbf{t}]_\times \mathbf{R}$ *of an antisymmetric matrix* $[\mathbf{t}]_\times$ *and a rotation matrix* $\mathbf{R}$ *if and only if*

$$det(\mathbf{E}) = 0 \tag{1}$$

$$f(\mathbf{E}) \equiv \frac{1}{2} Tr^2(\mathbf{E}\mathbf{E}^T) - Tr((\mathbf{E}\mathbf{E}^T)^2) = 0 \tag{2}$$

A characterization based upon algebraic geometry

- The Essential matrices "live" in a manifold of $\mathbb{P}^8$ defined by the two equations (1) and (2).
- Another definition of this manifold is the following:

Proposition 3 (Demazure) *The manifold $\mathcal{M}$ of complex essential matrices is characterized by the 9 polynomial equations of degree 3:*

$$\mathbf{F}(\mathbf{E}) \equiv \frac{1}{2} Tr(\mathbf{E}\mathbf{E}^T)\mathbf{E} - \mathbf{E}\mathbf{E}^T\mathbf{E} = 0 \tag{3}$$

Note that we include also *complex* Essential matrixes.

- The condition $det(\mathbf{E}) = 0$ is implied by (3).
- The contradiction between (1) and (2), and (3) is solved by the proposition:

Proposition 4 *The function* $\mathbf{F}$ *satisfies:*

$$Tr(\mathbf{F}(\mathbf{E})\mathbf{F}(\mathbf{E})^T) = -\frac{1}{2} Tr(\mathbf{E}\mathbf{E}^T)\, Tr(\mathbf{F}(\mathbf{E})\mathbf{E}^T) + 3det(\mathbf{E})^2 \tag{4}$$

The problem

- We investigate the question of determining the number of solutions to the motion problem for n point correspondences. We ask and answer two questions:

  1. What is the minimum value of n?
  2. For this minimum value, how many different displacements are there that are compatible with the matches?

The problem (*continued*)

- Since the dimensionality of the problem is 5, namely 3 parameters for the rotation and 2 for the translation, which is defined up to a scale factor, and since we obtain one equation for each correspondence, the minimum number of correspondences is 5.
- We show that when $n = 5$ the maximum number of solutions is 10.
- We can trace this problem back to Chasles (1855): "We are given, in a plane, two sets of seven points, in correspondence. For each set, find a point such that the two sets of seven lines they define are related by a collineation."

The problem (*continued*)

- We saw a very simple solution to that problem in lecture 6.
- The Euclidean version of Chasles problem:
  "We are given, in a plane, two sets of five points, in correspondence. For each set, find a point such that the two sets of seven lines (the five they define and the two defined by the cyclic points) are related by a collineation,"
  was solved by Erwin Kruppa in 1913.

General idea

- The basic idea is to enforce the fact that the two camera positions are obtained through a rigid displacement.
- Given two 3D points M and P with images (m, p) (resp. (m', p')) in the first image (resp. in the second), we note that the plane xCm (resp. xCp) is identical to the plane $x'C'm'$ (resp. $x'C'p'$).
- The angle between the planes xCm and xCp is therefore equal to the angle between the planes $x'C'm'$ and $x'C'p'$:

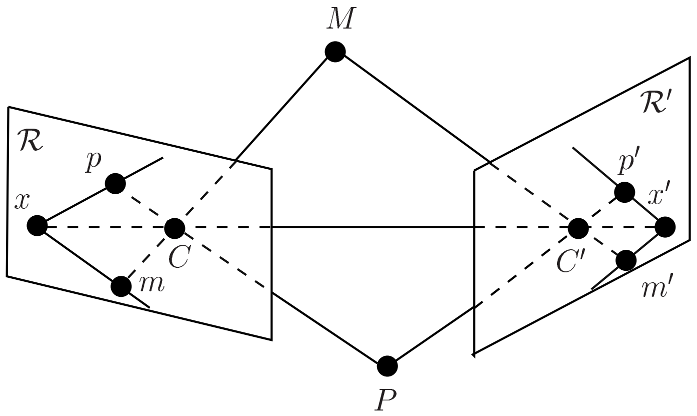

Figure 2: The plane angles $(xCm,\ xCp)$ and $(x'C'm',\ x'C'p')$ are equal.

General idea (*continued*)

- Introducing the absolute conic Ω, its two images ω and ω', the tangents u and v (resp. u' and v') from x (resp. x') to ω (resp. to ω') this is equivalent to saying that:

$$\{\langle x,\, m\rangle,\, \langle x,\, p\rangle;\, u,\, v\} = \{\langle x',\, m'\rangle,\, \langle x',\, p'\rangle;\, u',\, v'\}$$

- This is in turn equivalent to the fact that the pair (u, v) corresponds to the pair (u', v') in the epipolar collineation:

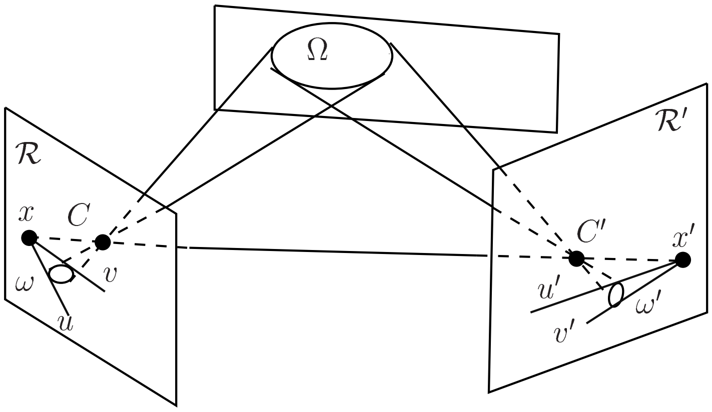

Figure 3: The images of the absolute conic Ω.

- It turns out that the precise correspondence between the two pairs of tangents is immaterial.

The algorithm

- Let $a\overline{\wedge}a'$, $b\overline{\wedge}b'$, $c\overline{\wedge}c'$, $d\overline{\wedge}d'$, $e\overline{\wedge}e'$ be 5 pairs of corresponding image points, and let x, x' be the unknown epipoles.
- Let l_a, l_b, l_c, l_d, l_e (resp. $l'_a, l'_b, l'_c, l'_d, l'_e$) be the epipolar lines.
- Assume that we have chosen projective coordinates such that (a, b, c, d) (resp. (a', b', c', d')) are the canonical projective bases of $\mathcal{R}$ and $\mathcal{R}'$.
- Let $\mathbf{B}$ (resp. $\mathbf{B}'$) be the matrix of ω (resp. ω').
- We parameterize the pencil of epipolar lines with its inersection with the line $\langle a, b\rangle$ (resp. $\langle a', b'\rangle$:

The algorithm (*continued*)

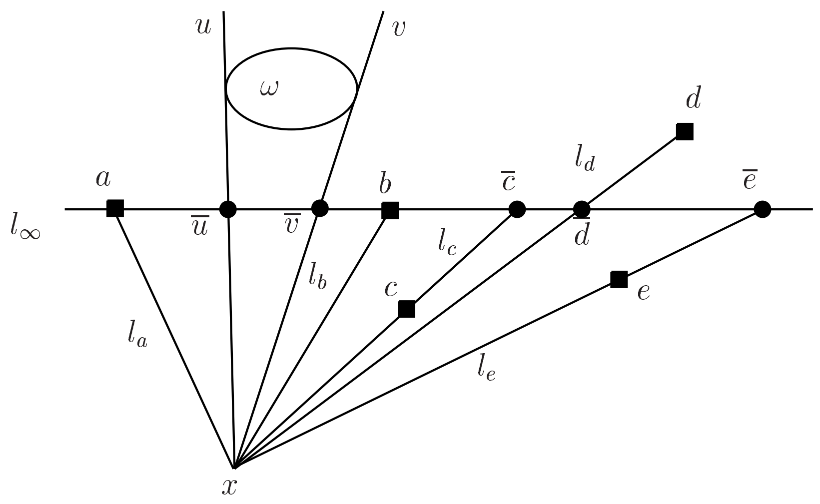

Figure 4: The situation in the image plane at the first time instant.

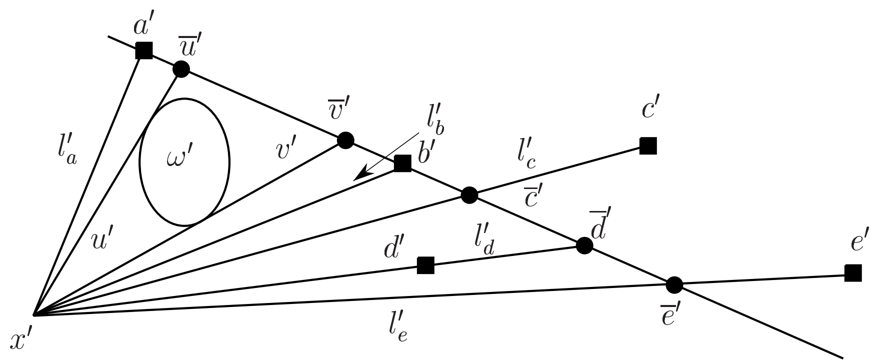

Figure 5: The situation in the image plane at the second time instant.

The algorithm (*continued*)

- The coordinates of $\mathbf{a}, \mathbf{b}, \bar{\mathbf{c}}, \bar{\mathbf{d}}, \bar{\mathbf{e}}$ as points of $\langle a, b\rangle$ are given by

$$\mathbf{a} = [1, 0]^T \qquad \mathbf{b} = [0, 1]^T$$
$$\bar{\mathbf{c}} = [x_1, x_2]^T \qquad \bar{\mathbf{d}} = [x_1 - x_3, x_2 - x_3]^T$$
$$\bar{\mathbf{e}} = [x_1 e_3 - x_3 e_1, x_2 e_3 - x_3 e_2]^T$$

- In order to compute the coordinates of $\bar{u}$ and $\bar{v}$ we use duality:

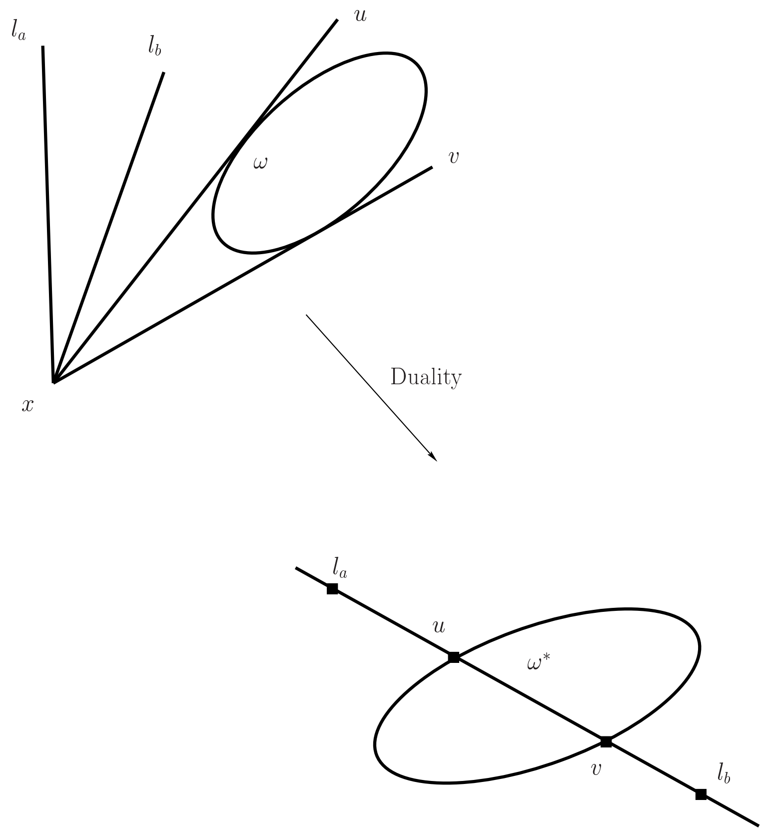

Figure 6: Duality.

The algorithm (*continued*)

- We write $\mathbf{u}$ and $\mathbf{v}$ as linear combinations of $\mathbf{l}_a$ and $\mathbf{l}_b$, e.g:

$$\mathbf{u} = \mathbf{l}_a + \theta \mathbf{l}_b$$

- The two values of θ corresponding to u and v are the roots of:

$$S(\mathbf{l}_b)\theta^2 + 2S(\mathbf{l}_a, \mathbf{l}_b)\theta + S(\mathbf{l}_a) = 0$$

- The points $\bar{u}$ and $\bar{v}$ have coordinates $[1, \theta]^T$.
- Let $\mathbf{K}$ be the matrix of ω^*, we define:

$$S(\mathbf{l}) = \mathbf{l}^T \mathbf{K} \mathbf{l} \quad S(\mathbf{l}, \mathbf{m}) = \mathbf{l}^T \mathbf{K} \mathbf{m} = \mathbf{m}^T \mathbf{K} \mathbf{l}$$

- The coefficients $S(\mathbf{l}_a), S(\mathbf{l}_a, \mathbf{l}_b)$ and $S(\mathbf{l}_b)$ are given by:

$$\begin{aligned} S(\mathbf{l}_a) &= -K_{33}x_2^2 - K_{22}x_3^2 + 2K_{23}x_2x_3 \\ S(\mathbf{l}_a, \mathbf{l}_b) &= K_{12}x_3^2 + K_{33}x_1x_2 - K_{23}x_1x_3 - K_{13}x_2x_3 \\ S(\mathbf{l}_b) &= -K_{33}x_1^2 - K_{11}x_3^2 + 2K_{13}x_1x_3 \end{aligned}$$

- similar expressions are obtained for the terms in $\mathcal{R}'$.

Writing the equations

- The collineations between the two pencils of epipolar lines induces a collineation between the lines $\langle a, b\rangle$ and $\langle a', b'\rangle$.
- Let $\mathbf{H}$ be its matrix.
- Since $\mathbf{a} \overline{\wedge} \mathbf{a}'$ and $\mathbf{b} \overline{\wedge} \mathbf{b}'$:

$$\mathbf{H} \simeq \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Applying H to the points $\bar{c}, \bar{d}$ and $\bar{e}$:

$$\begin{cases} \lambda x_1' & = & \alpha x_1 \\ \lambda x_2' & = & \beta x_2 \end{cases} \qquad \begin{cases} \mu(x_1' - x_3') & = & \alpha(x_1 - x_3) \\ \mu(x_2' - x_3') & = & \beta(x_2 - x_3) \end{cases}$$

$$\begin{cases} \nu(x_1' e_3' - x_3' e_1') & = & \alpha(x_1 e_3 - x_3 e_1) \\ \nu(x_2' e_3' - x_3' e_2') & = & \beta(x_2 e_3 - x_3 e_2) \end{cases}$$

where $\lambda,\ \mu,\ \nu$ are nonzero unknowns.

Writing the equations (*continued*)

- Eliminate $\alpha, \beta, \lambda, \mu$ and ν:

$$\frac{x'_1 - x'_3}{x'_2 - x'_3} \times \frac{x'_2}{x'_1} = \frac{x_1 - x_3}{x_2 - x_3} \times \frac{x_2}{x_1}$$

$$\frac{x'_1 e'_3 - x'_3 e'_1}{x'_2 e'_3 - x'_3 e'_2} \times \frac{x'_2}{x'_1} = \frac{x_1 e_3 - x_3 e_1}{x_2 e_3 - x_3 e_2} \times \frac{x_2}{x_1}$$

- The coordinates of $\bar{u}'$ and $\bar{v}'$ satisfy the equation:

$$S'(\mathbf{l}'_b)\theta'^2 + 2S'(\mathbf{l}'_a, \mathbf{l}'_b)\theta' + S'(\mathbf{l}'_a) = 0$$

- The relation between θ and θ' is:

$$\theta' = \frac{\beta}{\alpha}\theta$$

- Hence θ must satisfy the two equations:

$$\beta^2 S'(\mathbf{l}'_b)\theta^2 + 2\alpha\beta S'(\mathbf{l}'_a, \mathbf{l}'_b)\theta + \alpha^2 S'(\mathbf{l}'_a) = 0$$
$$S(\mathbf{l}_b)\theta^2 + 2S(\mathbf{l}_a, \mathbf{l}_b)\theta + S(\mathbf{l}_a) = 0$$

- The coefficients must be proportional:

$$\sigma S(\mathbf{l}_b) = \beta^2 S'(\mathbf{l}'_b) \quad \sigma S(\mathbf{l}_a, \mathbf{l}_b) = \alpha\beta S'(\mathbf{l}'_a, \mathbf{l}'_b)$$
$$\sigma S(\mathbf{l}_a) = \alpha^2 S'(\mathbf{l}'_a)$$

Writing the equations (*continued*)

- Eliminate σ, α, β:

$$\frac{S(\mathbf{l}_a)}{S(\mathbf{l}_a,\mathbf{l}_b)} \times \frac{x_1}{x_2} = \frac{S'(\mathbf{l}'_a)}{S'(\mathbf{l}'_a,\mathbf{l}'_b)} \times \frac{x'_1}{x'_2}$$
$$\frac{S(\mathbf{l}_b)}{S(\mathbf{l}_a,\mathbf{l}_b)} \times \frac{x_2}{x_1} = \frac{S'(\mathbf{l}'_b)}{S'(\mathbf{l}'_a,\mathbf{l}'_b)} \times \frac{x'_2}{x'_1}$$

- Apply the following changes of variables:

$$\mathbf{x} = \Phi\mathbf{u} \equiv (u_2u_3, u_3u_1, u_1u_2)^T$$
$$\mathbf{x}' = \Phi'\mathbf{u}' \equiv (u'_2u'_3, u'_3u'_1, u'_1u'_2)^T$$

- Φ (resp.Φ') are *quadratic* transformations. They are self-inverse since $\Phi(\Phi(\mathbf{u})) \simeq \mathbf{u}$. There exist 3 points, the *fundamental points* for which the transformation is not defined.

Writing the equations (*continued*)

- After cancelling $u_1u_2u_3$ and $u'_1u'_2u'_3$:

$$\frac{u_3-u_1}{u_3-u_2}=\frac{u'_3-u'_1}{u'_3-u'_2}$$
$$\frac{u_3e_3-u_1e_1}{u_3e_3-u_2e_2}=\frac{u'_3e'_3-u'_1e'_1}{u'_3e'_3-u'_2e'_2}$$

- After cancelling $u_1^2u_2u_3$ and $u_1'^2u'_2u'_3$:

$$\frac{-K_{33}u_3^2-K_{22}u_2^2+2K_{23}u_3u_2}{K_{12}u_1u_2+K_{33}u_3^2-K_{23}u_2u_3-K_{13}u_1u_3}=\frac{-K'_{33}u_3'^2-K'_{22}u_2'^2+2K'_{23}u'_3u'_2}{K'_{12}u'_1u'_2+K'_{33}u_3'^2-K'_{23}u'_2u'_3-K'_{13}u'_1u'_3}$$

- After cancelling $u_1u_2^2u_3$ and $u'_1u_2'^2u'_3$:

$$\frac{-K_{33}u_3^2-K_{11}u_1^2+2K_{13}u_1u_3}{K_{12}u_1u_2+K_{33}u_3^2-K_{23}u_2u_3-K_{13}u_1u_3}=\frac{-K'_{33}u_3'^2-K'_{11}u_1'^2+2K'_{13}u'_1u'_3}{K'_{12}u'_1u'_2+K'_{33}u_3'^2-K'_{23}u'_2u'_3-K'_{13}u'_1u'_3}$$

Writing the equations (*continued*)

- To eliminate $\mathbf{u}'$ we notice that the first two relations define a transformation from the plane $\mathbf{u}$ to the plane $\mathbf{u}'$:

$$\mathbf{u}' = \Sigma\mathbf{u} = \begin{bmatrix} e'_2[eu]_2[1u]_3 - e'_3[eu]_3[1u]_2 \\ e'_3[eu]_3[1u]_1 - e'_1[eu]_1[1u]_3 \\ e'_1[eu]_1[1u]_2 - e'_2[eu]_2[1u]_1 \end{bmatrix}$$

- Where:

$$[eu]_1 = e_2u_2 - e_3u_3 \qquad [1u]_1 = u_2 - u_3$$

  the other terms are obtained by circular permutations.

- Σ is, like Φ and Φ', a quadratic transformation.

- We obtain two homogeneous polynomial equations of degree 6 in u_1, u_2, u_3, which we denote by $A(\mathbf{u}) = 0$, $B(\mathbf{u}) = 0$.

- Each equation represents a sextic curve in the plane (u_1, u_2, u_3), and the solutions for $\mathbf{u} = \Phi^{-1}\mathbf{x} = \Phi\mathbf{x}$ are among the points of intersection of these two sextics. We call them $\mathcal{A}$ and $\mathcal{B}$.

A little bit of algebraic curves theory

- A plane curve is given by the zeros of a homogeneous polynomial equation in three variables.
- Let $f(\mathbf{x})$ be a polynomial describing a plane curve. Then $f(\mathbf{x})$ has a Taylor expansion

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \mathbf{h}.\nabla f(\mathbf{a}) + \dots$$

  at each point $\mathbf{a}$.
- If the terms of the Taylor expansion vanish up to and including the rth term then $f(\mathbf{x})$ is said to have order r at $\mathbf{a}$. Thus $\mathbf{a}$ is a point of $f(\mathbf{x})$ if and only if $r \geq 1$. If $r \geq 2$ then $f(\mathbf{x})$ is said to have a singularity of order r at $\mathbf{a}$.
- Bezout's theorem states that if $f(\mathbf{x})$, $g(\mathbf{x})$ are plane curves of degrees m and n respectively such that neither is contained in the other, then $f(\mathbf{x})$ and $g(\mathbf{x})$ intersect at exactly mn points.

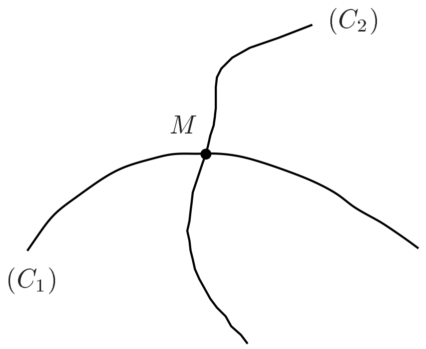

Figure 7: Intersection of two curves at a regular point.

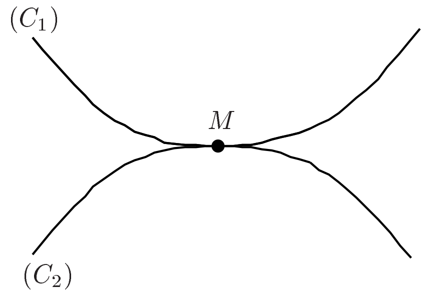

Figure 8: Two curves tangent at regular points.

A little bit of algebraic curves theory (*continued*)

- It is important to count the number of intersections correctly. For example, if $f(\mathbf{x})$ and $g(\mathbf{x})$ are tangent at a point then this counts 2 towards the total of mn intersections. If $f(\mathbf{x})$ has a singularity of order h at a point, and $g(\mathbf{x})$ has a singularity of order k at the same point, then this counts hk towards the total of intersections.

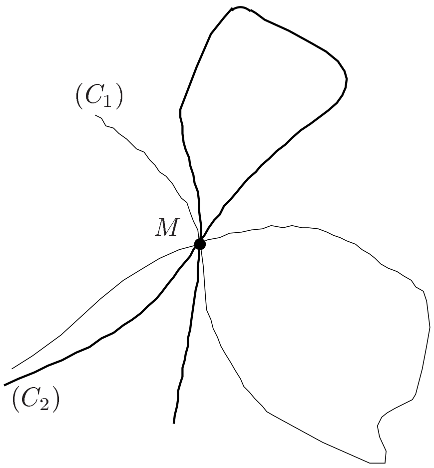

Figure 9: Intersection of two curves at points of order 2.

Eliminating impossible solutions

- There are three sources of impossible solutions:

1. Those that make the products $u_1 u_2 u_3$ and $u'_1 u'_2 u'_3$ equal to 0,
2. the points $\mathbf{m}$ such that $\Sigma \mathbf{m} = [0, 0, 0]^T$. Such points are called the fundamental points of Σ.
3. if we denote by D and D' the denominators of the left- and right-hand sides of the last two equations, then the points of intersection of the curves of equations $D(\mathbf{u}) = 0$ and $D'(\Sigma \mathbf{u}) = 0$ are impossible solutions.

Case 1

$$
\begin{array}{cccc}
\mathbf{u} & [1,0,0]^T & [0,1,0]^T & [0,0,1]^T \\
\mathbf{x} & [0,1,1]^T & [1,0,1]^T & [1,1,0]^T \\
\mathcal{A} & 3 & 1 & 1 \\
\mathcal{B} & 1 & 3 & 1
\end{array}
$$

We eliminate $3 \times 1 + 3 \times 1 + 1 \times 1 = 7$ solutions.

Case 2

Write:

$$\Sigma\mathbf{u} = \mathbf{Yu} \times \mathbf{Xu}$$

with:

$$\mathbf{X} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 0 & e_1'e_2 & -e_1'e_3 \\ -e_2'e_1 & 0 & e_2'e_3 \\ e_3'e_1 & -e_3'e_2 & 0 \end{bmatrix}$$

The fundamental points are therefore those points $\mathbf{u}$ such that $\mathbf{Yu} = 0$ or $\mathbf{Xu} = 0$, or such that $\mathbf{Xu}$ and $\mathbf{Yu}$ are parallel.

Eliminating impossible solutions (*continued*)

The three points are found to be d, e and another point p with the following orders:

$$\begin{array}{cccc} & \mathbf{d} & \mathbf{e} & \mathbf{p} \\ \mathcal{A} & 2 & 2 & 2 \\ \mathcal{B} & 2 & 2 & 2 \end{array}$$

We eliminate 12 solutions.

- **Case 3**

  The curve of equation $D(\mathbf{u}) = 0$ is a conic, and $D'(\Sigma\mathbf{u}) = 0$ is a quartic, since $D'(\mathbf{u}')$ is of degree 2 in $\mathbf{u}'$ and $\mathbf{u}' = \Sigma\mathbf{u}$ is of degree 2 in $\mathbf{u}$.

  These two curves have in general 8 intersections among which are included, as can be verified, the points $[1, 0, 0]^T$ and $[0, 1, 0]^T$ which have already been removed in case 1.

  Since each of the remaining 6 points is of order 1 on $\mathcal{A}$ and $\mathcal{B}$, we have obtained six more solutions to eliminate.

Eliminating impossible solutions (*continued*)

- There remain 11 intersections of $\mathcal{A}$ and $\mathcal{B}$ to consider.

- **Kruppa's spurious solution**
  It turns out that $\mathcal{A}$ and $\mathcal{B}$ not only intersect at $[0, 0, 1]^T$, but that they are tangential there, since the coefficients of u_3^5 in $A(\mathbf{u})$ and $B(\mathbf{u})$ are equal.

- It follows that $[0, 0, 1]^T$ must count twice in the intersection and not once.

- The number of solutions is reduced from 36 to 10, real or complex, for $\mathbf{u}$. The x-solutions are obtained as $\mathbf{x} = \Phi\mathbf{u}$ and the x'-solutions as $\mathbf{x}' = \Phi'\Sigma\mathbf{u}$.

The problem

- Are there any configurations of points such that there are always several solutions to the motion from points problem, even if an arbitrary large number of correspondences is available?
- The answer to this question is yes, there exist such sets of degenerate points and, for points in such a configuration, there exist at most three distinct camera displacements that are compatible with the correspondences.
- The problem has been studied in detail in the early 20th century by photogrammeters, mostly germans and austrians. The sets of degenerate points were called "gefährliche Flächen", dangerous surfaces.

Degenerate points lie on a quadric

- Assume that there exist two distinct camera motions $(\mathbf{R}, \mathbf{t})$ and $(\mathbf{R}', \mathbf{t}')$ such that the Longuet-Higgins equation is true for an *arbitrary* number of pairs of correspondences $\mathbf{m}$, $\mathbf{m}'$.
- Let C, C' and C'' be the three optical centers:

$$\mathbf{Q}_{\mathcal{F}_C}^{\mathcal{F}_{C'}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \quad \mathbf{Q}_{\mathcal{F}_C}^{\mathcal{F}_{C''}} = \begin{bmatrix} \mathbf{R}' & \mathbf{t}' \\ \mathbf{0} & 1 \end{bmatrix}$$

- Each camera motion corresponds to a *different* reconstruction and each reconstructed object is a quadric.
- Denote by $\mathbf{M}_{1 \backslash \mathcal{F}_C}$ and $\mathbf{M}_{2 \backslash \mathcal{F}_C}$ the two coordinate vectors, expressed in the normalized coordinate system of the first camera, corresponding to the reconstructed points whose image is m.
- The points M_1 and M_2 are on the optical ray $\langle C, m \rangle$.

Degenerate points lie on a quadric (*continued*)

- Let $\mathbf{M}_{1\backslash\mathcal{F}_{C'}}$ and $\mathbf{M}_{2\backslash\mathcal{F}_{C''}}$ be the corresponding coordinate vectors of the same points expressed in the normalized camera after the first and second motion, respectively. The points M_1 and M_2 are on the optical rays $\langle C', m' \rangle$ and $\langle C'', m'' \rangle$, respectively.

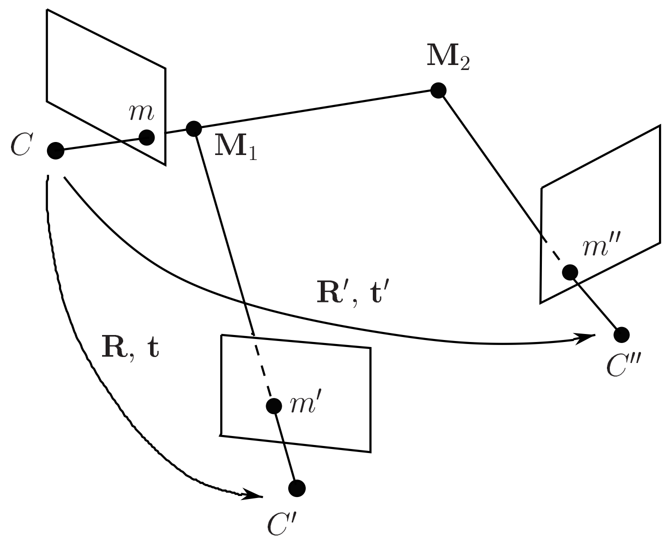

Figure 10: Two motions yielding the same images.

- Write the Longuet-Higgins equation for the first camera motion as:

$$\mathbf{m}' \cdot (\mathbf{t} \times \mathbf{Rm}) = 0$$

Degenerate points lie on a quadric (*continued*)

- By definition of the ambiguity, $\mathbf{m}' = \mathbf{m}''$, where the two vectors are expressed in the local coordinate systems attached to the camera after the two motions.
- The previous equation can be rewritten as:

$$\mathbf{M}_{2\backslash\mathcal{F}_{C''}} \cdot (\mathbf{t} \times \mathbf{R}\mathbf{M}_{2\backslash\mathcal{F}_C}) = 0$$

Degenerate points lie on a quadric (*continued*)

- Use the fact that $\mathbf{M}_{2\backslash\mathcal{F}_C} = \mathbf{R}'^T(\mathbf{M}_{2\backslash\mathcal{F}_{C''}} - \mathbf{t}')$, we obtain:

$$\mathbf{M}^T_{2\backslash\mathcal{F}_{C''}}\mathbf{E}\mathbf{R}'^T(\mathbf{M}_{2\backslash\mathcal{F}_{C''}} - \mathbf{t}') = 0$$

- This is the equation of a quadric surface going through the points $\mathbf{M}_{2\backslash\mathcal{F}_{C''}} = \mathbf{0}$ and $\mathbf{M}_{2\backslash\mathcal{F}_{C''}} = \mathbf{t}'$, i.e., C'' and C. It also contains the point $\mathbf{M}_{2\backslash\mathcal{F}_{C''}} = \mathbf{t}' + \mathbf{R}'\mathbf{R}^T\mathbf{t}$, i.e. the point C'.

- This quadric is the locus of the ambiguous points as reconstructed from the second camera motion. It is called a Maybank quadric.

Degenerate points lie on a quadric (*continued*)

- Exchange the roles of the first and the second camera motions and obtain the equation of another quadric:

$$\mathbf{M}_{1\backslash\mathcal{F}_{C'}}^T \mathbf{E}'\mathbf{R}^T(\mathbf{M}_{1\backslash\mathcal{F}_{C'}} - \mathbf{t}) = 0,$$

  which is the locus of the ambiguous points as reconstructed from the first camera motion.

- Note that the quadric goes through the points $\mathbf{M}_{1\backslash\mathcal{F}_{C'}} = \mathbf{0}$ and $\mathbf{M}_{1\backslash\mathcal{F}_{C'}} = \mathbf{t}$, i.e., C' and C. It also contains the point $\mathbf{M}_{1\backslash\mathcal{F}_{C'}} = \mathbf{t} - \mathbf{R}\mathbf{R}'^T\mathbf{t}'$, i.e. the point C''.

How many ambiguous motions?

- The question is the same as asking how many essential matrices $\mathbf{E}_i$ can satisfy:

$$\mathbf{m}'^T \mathbf{E}_i \mathbf{m} = 0$$

for pairs of corresponding points $\mathbf{m} \leftrightarrow \mathbf{m}'$ where $\mathbf{m}$, $\mathbf{m}'$ range over open subsets of the retinal planes.

- Note that for $i = 2$ this implies the existence of a quadratic transformation between the two images:

$$\mathbf{m}' \simeq \mathbf{E}_1 \mathbf{m} \times \mathbf{E}_2 \mathbf{m}$$

- We need the following proposition:

Proposition 5 (Maybank) *Let* $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ *be essential matrices such that* $\mathbf{m}^T \mathbf{E}_i \mathbf{m}' = 0$ *for i = 1,2,3, where* $\mathbf{m} \leftrightarrow \mathbf{m}'$, *and* $\mathbf{m}$, $\mathbf{m}'$ *both range over open subsets of the retinal planes modeled as two projective planes* $\mathcal{P}^2$. *Then* $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ *are linearly dependent.*

How many ambiguous motions? (*continued*)

- The answer to the original question lies in the following theorem:

Theorem 2 *If the 3D points lie on a Maybank quadric, there may be as many as three distinct camera motions that yield the same image of those points.*

Proof : The previous proposition tells us that all those matrices are on a projective line of $\mathcal{P}^8$. Therefore, the question is equivalent to asking how many intersection points a general line can have with $\mathcal{M}$, the manifold of essential matrices. Since we saw that the equations of $\mathcal{M}$ are of degree 3, the number of intersections of a line with $\mathcal{M}$ is at most 3. $\square$

The problem

- Unlike the case for points, matching two lines in two frames as being the images of the same 3D line does not constrain the rigid displacement between those frames: we need at least three frames.

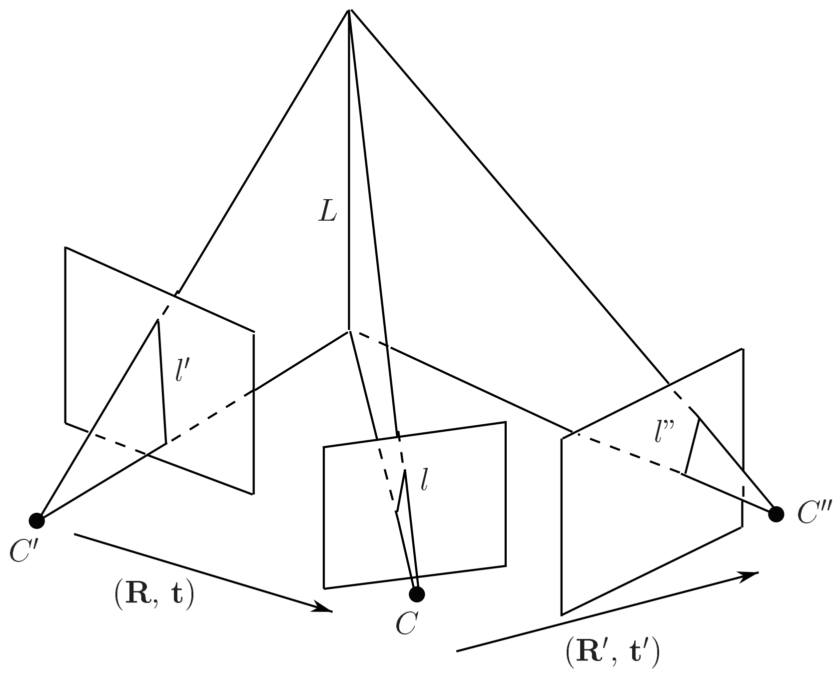

Figure 11: Three frames provide constraints about motion

The equations

- Let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be the three perspective projection matrices of the three cameras.
- Let l, l' and l'' be the three images of the 3D line L.
- The 3D line L is the meet of the two planes (C', l') and (C'', l''):

$$\mathbf{L} = \mathcal{P}'^T \mathbf{l}' \triangle \mathcal{P}''^T \mathbf{l}''$$

- Its image in the first retina is:

$$\tilde{\mathcal{P}}\mathbf{L}$$

- The *two* motion equations are:

$$\mathbf{l} \simeq \tilde{\mathcal{P}}\mathbf{L} = \tilde{\mathcal{P}}(\mathcal{P}'^T \mathbf{l}' \triangle \mathcal{P}''^T \mathbf{l}'')$$

The equations (*continued*)

- After some algebra (we 'll come back to it in a future lecture) we find:

$$\mathbf{l} \simeq \begin{bmatrix} \mathbf{l}'^T \mathbf{G}^1 \mathbf{l}'' \\ \mathbf{l}'^T \mathbf{G}^2 \mathbf{l}'' \\ \mathbf{l}'^T \mathbf{G}^3 \mathbf{l}'' \end{bmatrix}$$

- The correspondence $(l', l'') \longrightarrow l$ is governed by the *trifocal tensor*.
- The three 3×3 matrices $\mathbf{G}^n$ are equal to:

$$\mathbf{G}^n = \mathbf{t}\mathbf{R}''^{(n)T} + \mathbf{R}'^{(n)}\mathbf{t}'^T$$

- A correspondence of three lines is equivalent to the *two* equations:

$$l_1 : l_2 : l_3 = \mathbf{l}'^T \mathbf{G}^1 \mathbf{l}'' : \mathbf{l}'^T \mathbf{G}^2 \mathbf{l}'' : \mathbf{l}'^T \mathbf{G}^3 \mathbf{l}''$$

The equations (*continued*)

- Since we have 11 motion parameters, $5 + \frac{1}{2}$ line correspondences are necessary to compute them.
- As of today there is no equivalent for lines of the five points algorithm.
- The set of ambiguous lines is known.

The Liu and Huang equations

- The Liu and Huang equations decouple rotation and translation:

$$\mathbf{l} \cdot (\mathbf{R}^T \mathbf{l}' \times \mathbf{R}'^T \mathbf{l}'') = 0$$
$$| \mathbf{t}^T \mathbf{l}' | \, \| \mathbf{l} \wedge \mathbf{R}'^T \mathbf{l}'' \| = | \mathbf{t}''^T \mathbf{l}'' | \, \| \mathbf{l} \wedge \mathbf{R}^T \mathbf{l}' \|$$

- The previous equations can be written:

$$\mathbf{l} \times (a \tilde{\mathbf{l}}' + b \tilde{\mathbf{l}}'')$$

where:

$$\tilde{\mathbf{l}}' = \mathbf{R}^T \mathbf{l}' \quad \tilde{\mathbf{l}}'' = \mathbf{R}'^T \mathbf{l}''$$
$$a = \mathbf{l}''^T \mathbf{t}' \quad b = \mathbf{l}'^T \mathbf{t}$$

- Therefore we have:

$$| \, \mathbf{l} \, \tilde{\mathbf{l}}' \, \tilde{\mathbf{l}}'' \, |$$
$$| \, a \, | \; \| \mathbf{l} \times \tilde{\mathbf{l}}' \| = | \, b \, | \; \| \mathbf{l} \times \tilde{\mathbf{l}}'' \|$$

These are the Liu and Huang equations.