Second problem set for 6.965

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We note $(\varepsilon_i, i = 1, \cdots, n)$ the canonical (in particular unimodal) basis of $\mathbb{R}^n$. We also note $\mathcal{A}_n$ the affine space of dimension n and $\mathbb{P}^n$ its projective completion. If a point M of $\mathcal{A}_n$ has affine coordinates $(X_1, \ldots, X_n)$, it has projective coordinates $(X_1, \ldots, X_n, 1)$ in $\mathbb{P}^n$. The equation of the hyperplane at infinity of $\mathcal{A}_n$ is therefore $x_{n+1} = 0$ if $(x_1, \ldots, x_{n+1})$ denote the projective coordinates.

1. **Planar case**

   (a) What is the associated cobasis $(*\varepsilon_1, *\varepsilon_2, *\varepsilon_3)$ of the canonical basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$? Justify your answer.

   (b) We consider the affine plane $\mathcal{A}_2$ and its projective completion $\mathbb{P}^2$. Let m_1 (respectively m_2) be the affine point of coordinates $(1, 0)$ (resp. $(0, 1)$). Write the affine equation of the line (m_1, m_2). What is its projective equation?

   (c) Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two projective coordinate vectors of the two projective points m_1 and m_2. Compute the coordinates of $\mathbf{m}_1 \nabla \mathbf{m}_2$ in the canonical basis of $G_2(\mathbb{R}^3)$ and in the cobasis of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Any connection with the previous question?

   (d) Let l' be the line of affine equation $X_1 + X_2 = 2$. Give a coordinate vector of l' in the cobasis $(*\varepsilon_1, *\varepsilon_2, *\varepsilon_3)$. Compute the coordinates of the meet of l and l' in the canonical basis of $G_1(\mathbb{R}^3)$. Geometric interpretation?

2. **3D case: lines**

(a) What is the associated cobasis of the basis of $G_2(\mathbb{R}^4)$ $(\varepsilon_1 \nabla \varepsilon_2, \varepsilon_3 \nabla \varepsilon_1, \varepsilon_1 \nabla \varepsilon_4, \varepsilon_2 \nabla \varepsilon_3, \varepsilon_2 \nabla \varepsilon_4, \varepsilon_3 \nabla \varepsilon_4)$ (note that this is not the canonical basis of $G_2(\mathbb{R}^4)$)? Justify your answer.

(b) We consider the affine space $\mathcal{A}_3$ and its projective completion $\mathbb{P}^3$. Let M_1 (resp. M_2) be an affine point with affine coordinates (X_1, Y_1, Z_1) (resp. (X_2, Y_2, Z_2)). Express the Plücker coordinates of the line $L = (M_1, M_2)$ in the previous basis of $G_2(\mathbb{R}^4)$ in terms of the vectors $\overrightarrow{OM_1}$, $\overrightarrow{OM_2}$, and $\overrightarrow{M_1M_2}$, where O is the origin of the affine coordinate system.

(c) Write the Plücker relation for the line L. Geometric interpretation?

(d) What are the Plücker coordinates of a line in the plane at infinity?

(e) Given the Plücker coordinates of a line not in the plane at infinity, how do you recover an affine point on the line?

(f) Consider now the dual line $*L$ of L. Compute its Plücker coordinates in the previous basis of $G_2(\mathbb{R}^4)$. Geometric interpretation?

3. **3D case: planes**

(a) What is the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$? Justify your answer.

(b) Let M_1 (resp. M_2, M_3) the affine point of coordinates $(1, 0, 0)$ (resp. $(0, 1, 0)$, $(0, 0, 1)$). Write the affine equation of the plane (M_1, M_2, M_3). What is its projective equation?

(c) Let $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ be three projective coordinate vectors of the projective points M_1, M_2 and M_3. Compute the coordinates of $\mathbf{M}_1 \nabla \mathbf{M}_2 \nabla \mathbf{M}_3$ in the canonical basis of $G_3(\mathbb{R}^4)$ and in the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$. Any connection with the previous question?

(d) Consider the three planes Π_1, Π_2, Π_3 of affine equations $X_1 = 1$, $X_2 = 1$ and $X_3 = 1$, respectively. Give coordinate vectors of these three planes in the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$. Compute the coordinates of their point of intersection using the analysis in section 2.4.5 of the notes.

4. **3D case: planes and lines**

(a) Let L be a line defined as the meet of two planes Π_1 and Π_2: $L = \Pi_1 \vartriangle \Pi_2$.Compute the projective coordinates of the point of intersection $\Pi_0 \vartriangle L$ of L with Π_0.

(b) Check your answer with Π_1 defined by the equation $X_3 = 0$, Π_2 defined by the equation $X_2 = 0$, $A = [1, 0, 0, 0]^T$, $B = [0, 1, 0, 0]^T$ and $C = [0, 0, 1, 0]^T$. Geometric interpretation?

(c) Consider the two planes of equation $X_1 = X_2$ and $X_2 = X_3$. Compute the Plücker coordinates of their line of intersection. Geometric interpretation?

(d) Same question with the planes $X_1 = X_2$ and $X_1 = X_2 + 1$.