Second problem set for 6.965

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April 5, 2000

We note $(\varepsilon_i, i = 1, \cdots, n)$ the canonical (in particular unimodal) basis of $\mathbb{R}^n$. We also note $\mathcal{A}_n$ the affine space of dimension n and $\mathbb{P}^n$ its projective completion. If a point M of $\mathcal{A}_n$ has affine coordinates $(X_1, \ldots, X_n)$, it has projective coordinates $(X_1, \ldots, X_n, 1)$ in $\mathbb{P}^n$. The equation of the hyperplane at infinity of $\mathcal{A}_n$ is therefore $x_{n+1} = 0$ if $(x_1, \ldots, x_{n+1})$ denote the projective coordinates.

1. **Planar case**

   (a) What is the associated cobasis $(*\varepsilon_1, *\varepsilon_2, *\varepsilon_3)$ of the canonical basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$? Justify your answer.

   Answer

   $$*\varepsilon_1 = \mid \varepsilon_1,\, \hat{\varepsilon_1},\, \varepsilon_2,\, \varepsilon_3 \mid^{-1} \varepsilon_2 \nabla \varepsilon_3 = \varepsilon_2 \nabla \varepsilon_3$$

   etc..

   $$*\varepsilon_2 = -\varepsilon_1 \nabla \varepsilon_3 \quad * \varepsilon_3 = \varepsilon_1 \nabla \varepsilon_2$$

   (b) We consider the affine plane $\mathcal{A}_2$ and its projective completion $\mathbb{P}^2$. Let m_1 (respectively m_2) be the affine point of coordinates $(1, 0)$ (resp. $(0, 1)$). Write the affine equation of the line (m_1, m_2). What is its projective equation?

   Answer
   Affine line equation:

   $$X_1 + X_2 = 1$$

   Projective line equation:

   $$x_1 + x_2 = x_3 \tag{1}$$

(c) Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two projective coordinate vectors of the two projective points m_1 and m_2. Compute the coordinates of $\mathbf{m}_1 \nabla \mathbf{m}_2$ in the canonical basis of $G_2(\mathbb{R}^3)$ and in the cobasis of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Any connection with the previous question?

Answer

$$\mathbf{m}_1 \nabla \mathbf{m}_2 = \varepsilon_1 \nabla \varepsilon_2 + \varepsilon_1 \nabla \varepsilon_3 - \varepsilon_2 \nabla \varepsilon_3 = - * \varepsilon_1 - *\varepsilon_2 + *\varepsilon_3$$

The coordinates of $\mathbf{m}_1 \nabla \mathbf{m}_2$ in the cobasis are the coordinates in the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of the cross-product of the vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ and are proportional to the coordinates $[1, 1, -1]^T$ of the line (m_1, m_2) in the cobasis, coordinates that can be read from equation (1).

(d) Let l' be the line of affine equation $X_1 + X_2 = 2$. Give a coordinate vector of l' in the cobasis $(*\varepsilon_1, *\varepsilon_2, *\varepsilon_3)$. Compute the coordinates of the meet of l and l' in the canonical basis of $G_1(\mathbb{R}^3)$. Geometric interpretation?

Answer

The projective equation of l' is $x_1 + x_2 = 2x_3$. l' is therefore represented in the cobasis by the vector $[1, 1, -2]^T$. We have $l = *\varepsilon_1 + *\varepsilon_2 - *\varepsilon_3$ and $l' = *\varepsilon_1 + *\varepsilon_2 - 2 * \varepsilon_3$. Some algebra shows that $l \bigtriangleup l' = -\varepsilon_1 + \varepsilon_2$. It represents the point of intersection of l and l'. It is on the line at infinity since the two affine lines l and l' are parallel.

2. **3D case: lines**

(a) What is the associated cobasis of the basis of $G_2(\mathbb{R}^4)$ $(\varepsilon_1 \nabla \varepsilon_2, \varepsilon_3 \nabla \varepsilon_1, \varepsilon_1 \nabla \varepsilon_4, \varepsilon_2 \nabla \varepsilon_3, \varepsilon_2 \nabla \varepsilon_4, \varepsilon_3 \nabla \varepsilon_4)$ (note that this is not the canonical basis of $G_2(\mathbb{R}^4)$)? Justify your answer.

Answer

By definition:

$$*(\varepsilon_1 \nabla \varepsilon_2) = \mid \varepsilon_1,\, \varepsilon_2,\, \varepsilon_3,\, \varepsilon_4 \mid^{-1} \varepsilon_3 \nabla \varepsilon_4$$

etc:

$$*(\varepsilon_3 \nabla \varepsilon_1) = \varepsilon_2 \nabla \varepsilon_4 *(\varepsilon_1 \nabla \varepsilon_4) = \varepsilon_2 \nabla \varepsilon_3 *(\varepsilon_2 \nabla \varepsilon_3) = \varepsilon_1 \nabla \varepsilon_4$$
$$*(\varepsilon_2 \nabla \varepsilon_4) = \varepsilon_3 \nabla \varepsilon_1 *(\varepsilon_3 \nabla \varepsilon_4) = \varepsilon_1 \nabla \varepsilon_2$$

(b) We consider the affine space $\mathcal{A}_3$ and its projective completion $\mathbb{P}^3$. Let M_1 (resp. M_2) be an affine point with affine coordinates (X_1, Y_1, Z_1) (resp. (X_2, Y_2, Z_2)). Express the Plücker coordinates of the line $L = (M_1, M_2)$ in the previous basis of $G_2(\mathbb{R}^4)$ in terms of the vectors $\overrightarrow{OM_1}$, $\overrightarrow{OM_2}$, and $\overrightarrow{M_1M_2}$, where O is the origin of the affine coordinate system.

Answer

Write $M_i = X_i\varepsilon_1 + Y_i\varepsilon_2 + Z_i\varepsilon_3 + \varepsilon_3$, $i = 1, 2$ and compute $M_1 \nabla M_2$ One obtains

$$(Y_1Z_2 - Y_2Z_1)\varepsilon_2\nabla\varepsilon_3 + (Z_1X_2 - Z_2X_1)\varepsilon_3\nabla\varepsilon_1 + (X_1Y_2 - X_2Y_1)\varepsilon_1\nabla\varepsilon_2 + \\ (X_1 - X_2)\varepsilon_1\nabla\varepsilon_4 + (Y_1 - Y_2)\varepsilon_2\nabla\varepsilon_4 + (Z_1 - Z_2)\varepsilon_3\nabla\varepsilon_4$$

The first three coordinates are those of the cross-product $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$, the other three those of $\overrightarrow{M_2M_1}$.

(c) Write the Plücker relation for the line L. Geometric interpretation?

Answer

In our basis, the Plücker relation is written:

$$L_{12}L_{34} + L_{31}L_{24} + L_{14}L_{23} = 0$$

which is exactly

$$\overrightarrow{M_2M_1} \cdot (\overrightarrow{OM_1} \times \overrightarrow{OM_2}) = 0$$

which says that the vector $\overrightarrow{M_2M_1}$ is orthogonal to the vector $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$ which is certainly true.

(d) What are the Plücker coordinates of a line in the plane at infinity?

Answer

A line in the plane at infinity is such that any two of its points are in the plane at infinity which shows that the coordinates along the vectors $\varepsilon_1\nabla\varepsilon_4$, $\varepsilon_2\nabla\varepsilon_4$ and $\varepsilon_3\nabla\varepsilon_4$ are all equal 0.

(e) Given the Plücker coordinates of a line not in the plane at infinity, how do you recover an affine point on the line?

Answer

First, according to question (b) above, the affine line is in the plane going through the origin O and normal to the vector $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$ which is obtained from three of the Plücker coordinates of the line. We only need to recover the vector from the origin to the line in that plane. If $\overrightarrow{OM_1} \times \overrightarrow{OM_2} = \mathbf{0}$, the line goes through the origin and we are done. Otherwise let o be the projection of O on the line and $\mathbf{d} = \overrightarrow{Oo}$. We need to revover $\mathbf{d}$. We write $\overrightarrow{OM_i} = \mathbf{d} + \overrightarrow{oM_i}$, $i = 1, 2$ and compute $\overrightarrow{OM_1} \times \overrightarrow{OM_2} = \mathbf{d} \times \overrightarrow{M_1M_2}$ which yields

$$\mathbf{d} = \frac{1}{\|\overrightarrow{M_1M_2}\|^2} \overrightarrow{M_1M_2} \times (\overrightarrow{OM_1} \times \overrightarrow{OM_2})$$

(f) Consider now the dual line $*L$ of L. Compute its Plücker coordinates in the previous basis of $G_2(\mathbb{R}^4)$. Geometric interpretation?

Answer

According to (b), the Plücker coordinates of L are those of the vector $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$ along the three vectors $\varepsilon_2 \nabla \varepsilon_3$, $\varepsilon_3 \nabla \varepsilon_1$, $\varepsilon_1 \nabla \varepsilon_2$ and those of the vector $\overrightarrow{M_2M_1}$ along the vectors $\varepsilon_1 \nabla \varepsilon_4$, $\varepsilon_2 \nabla \varepsilon_4$ and $\varepsilon_3 \nabla \varepsilon_4$. Using question (a), we see that the Plücker coordinates of $*L$ are those of the vector $\overrightarrow{M_2M_1}$ along the vectors $\varepsilon_2 \nabla \varepsilon_3$, $\varepsilon_3 \nabla \varepsilon_1$, $\varepsilon_1 \nabla \varepsilon_2$ and those of the vector $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$ along the vectors $\varepsilon_1 \nabla \varepsilon_4$, $\varepsilon_2 \nabla \varepsilon_4$ and $\varepsilon_3 \nabla \varepsilon_4$. Therefore, if L is in the plane at infinity, $*L$ goes through the origin O and is perpendicular to L. Otherwise, $*L$ is in the plane defined by O and the two vectors $\mathbf{d}$ and $\overrightarrow{OM_1} \times \overrightarrow{OM_2}$. In that plane, it is orthogonal to the vector $\mathbf{d}$. Let us call $\mathbf{d}^*$ the vector $\overrightarrow{Oo'}$ where o' is the projection of the origin O on $*L$. We have:

$$\mathbf{d}^* = \frac{1}{\|\overrightarrow{OM_1} \times \overrightarrow{OM_2}\|^2} \overrightarrow{M_1M_2} \times (\overrightarrow{OM_1} \times \overrightarrow{OM_2})$$

Which shows that the inner product $\mathbf{d} \cdot \mathbf{d}^*$ is equal to 1. As a special case, when L is at infinity $\|\mathbf{d}\| = \infty$ and hence $\mathbf{d}^* = \mathbf{0}$.

3. **3D case: planes**

(a) What is the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$? Justify your answer.

Answer

$$*\varepsilon_1 = \varepsilon_2 \nabla \varepsilon_3 \nabla \varepsilon_4 \quad *\varepsilon_2 = -\varepsilon_1 \nabla \varepsilon_3 \nabla \varepsilon_4$$
$$*\varepsilon_3 = \varepsilon_1 \nabla \varepsilon_2 \nabla \varepsilon_4 \quad *\varepsilon_4 = -\varepsilon_1 \nabla \varepsilon_2 \nabla \varepsilon_3$$

(b) Let M_1 (resp. M_2, M_3) the affine point of coordinates $(1, 0, 0)$ (resp. $(0, 1, 0)$, $(0, 0, 1)$). Write the affine equation of the plane (M_1, M_2, M_3). What is its projective equation?

Answer
Affine:

$$X_1 + X_2 + X_3 = 1$$

Projective:

$$x_1 + x_2 + x_3 = x_4$$

(c) Let $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ be three projective coordinate vectors of the projective points M_1, M_2 and M_3. Compute the coordinates of $\mathbf{M}_1 \triangledown \mathbf{M}_2 \triangledown \mathbf{M}_3$ in the canonical basis of $G_3(\mathbb{R}^4)$ and in the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$. Any connection with the previous question?

Answer
We write $\mathbf{M}_i = \varepsilon_i + \varepsilon_4$, $i = 1, 2, 3$ and compute the join of the three vectors:

$$\mathbf{M}_1 \triangledown \mathbf{M}_2 \triangledown \mathbf{M}_3 = \varepsilon_1 \nabla \varepsilon_2 \nabla \varepsilon_3 + \varepsilon_1 \nabla \varepsilon_2 \nabla \varepsilon_4 - \varepsilon_1 \nabla \varepsilon_3 \nabla \varepsilon_4 + \varepsilon_2 \nabla \varepsilon_3 \nabla \varepsilon_4 = \\ * \varepsilon_1 + *\varepsilon_2 + *\varepsilon_3 - *\varepsilon_4$$

The coordinates of $\mathbf{M}_1 \triangledown \mathbf{M}_2 \triangledown \mathbf{M}_3$ in the associated cobasis are proportional to the coefficients of the projective equation of the plane.

(d) Consider the three planes Π_1, Π_2, Π_3 of affine equations $X_1 = 1$, $X_2 = 1$ and $X_3 = 1$, respectively. Give coordinate vectors of these three planes in the associated cobasis of the canonical basis of $G_1(\mathbb{R}^4)$. Compute the coordinates of their point of intersection using the analysis in section 2.4.5 of the notes.

Answer
The coordinate vectors in the associated cobasis are proportional

to the coefficients of the projective equations of the three planes. Hence we find $[1, 0, 0, -1]^T$, $[0, 1, 0, -1]^T$ and $[0, 0, 1, -1]^T$. The coordinates of their point of intersection are the values of the 3×3 minors of the 4×3 matrix built with three vectors. We find the vector $[1, 1, 1, 1]^T$ which represents the affine point of coordinates $[1, 1, 1]^T$ intersection of our three planes.

4. **3D case: planes and lines**

   (a) Let L be a line defined as the meet of two planes Π_1 and Π_2: $L = \Pi_1 \vartriangle \Pi_2$.Compute the projective coordinates of the point of intersection $\Pi_0 \vartriangle L$ of L with Π_0, where Π_0 is defined by the three points A, B and C.

   Answer

$$\Pi_0 \vartriangle L = (A \triangledown B \triangledown C) \vartriangle (\Pi_1 \vartriangle \Pi_2) = ((A \triangledown B \triangledown C) \vartriangle \Pi_1) \vartriangle \Pi_2$$

because of the associativity of the meet.

$$(A \triangledown B \triangledown C) \vartriangle \Pi_1 = [A, \Pi_1] B \triangledown C - [B, \Pi_1] A \triangledown C + [C, \Pi_1] A \triangledown B$$

hence:

$$\begin{aligned}((A \triangledown B \triangledown C) \vartriangle \Pi_1) \vartriangle \Pi_2 = {} & [A, \Pi_1](B \triangledown C) \vartriangle \Pi_2 - \\ & [B, \Pi_1](A \triangledown C) \vartriangle \Pi_2 + [C, \Pi_1](A \triangledown B) \vartriangle \Pi_2\end{aligned}$$

Each term can then be expanded:

$$\begin{aligned}(B \triangledown C) \vartriangle \Pi_2 &= [B, \Pi_2] C - [C, \Pi_2] B \\ (A \triangledown C) \vartriangle \Pi_2 &= [A, \Pi_2] C - [C, \Pi_2] A \\ (A \triangledown B) \vartriangle \Pi_2 &= [A, \Pi_2] B - [B, \Pi_2] A\end{aligned}$$

We end up with the nice symmetric expression:

$$\begin{aligned}&([B, \Pi_1][C, \Pi_2] - [B, \Pi_2][C, \Pi_1]) A + \\ &\quad ([C, \Pi_1][A, \Pi_2] - [C, \Pi_2][A, \Pi_1]) B + \\ &\quad\quad ([A, \Pi_1][B, \Pi_2] - [A, \Pi_2][B, \Pi_1]) C\end{aligned}$$

(b) Check your answer with Π_1 defined by the equation $X_3 = 0$, Π_2 defined by the equation $X_2 = 0$, $A = [1, 0, 0, 0]^T$, $B = [0, 1, 0, 0]^T$ and $C = [0, 0, 1, 0]^T$. Geometric interpretation?

Answer

Plugging those coordinates in the formula obtained in the previous question we obtain $\Pi_0 \vartriangle L \simeq A$. Geometrically, the line L is the X_1 axis of the affine basis and Π_0 is the plane at infinity. Therefore their point of intersection is the point at infinity of L which is A.

(c) Consider the two planes of equation $X_1 = X_2$ and $X_2 = X_3$. Compute the Plücker coordinates of their line of intersection. Geometric interpretation?

Answer

Write $\Pi_1 = *\varepsilon_1 - *\varepsilon_2$ and $\Pi_2 = *\varepsilon_2 - *\varepsilon_3$. The line of intersection is obtained as $\Pi_1 \vartriangle \Pi_2$. Using the usual artillary we obtain $\Pi_1 \vartriangle \Pi_2 = *(\varepsilon_1 \nabla \varepsilon_2) - *(\varepsilon_1 \nabla \varepsilon_3) + *(\varepsilon_2 \nabla \varepsilon_3) = \varepsilon_1 \nabla \varepsilon_4 + \varepsilon_2 \nabla \varepsilon_4 + \varepsilon_3 \nabla \varepsilon_4$. From question (e) of part 2 we deduce that the line goes through the origin and is parallel to the vector $[1, 1, 1]^T$.

(d) Same question with the planes $X_1 = X_2$ and $X_1 = X_2 + 1$.

Answer

The same computation yields $\Pi_1 \vartriangle \Pi_2 = -\varepsilon_2 \nabla \varepsilon_3 + \varepsilon_3 \nabla \varepsilon_1$ which says that this line is in the plane at infinity, at the intersection with the planes perpendicular to the vector $[-1, 1, 0]^T$.