Third problem set for 6.965

Olivier Faugeras

April 4, 2000

1. Show that the transpose $\boldsymbol{\mathcal{P}}^T$ of the perspective projection matrix can be interpreted as a mapping from the set of lines of $\mathbb{P}^2$ to the set of planes of $\mathbb{P}^3$. What is the image of the line l represented by $\mathbf{l}$, as a function of the three projection planes?

2. This problem investigates a curious property of vanishing points. We know that parallel lines intersect at a point situated in Π_∞ for lines of $\mathcal{P}^3$, and on l_∞ for $\mathcal{P}^2$. This property of intersecting at a point is preserved by perspective projection and therefore the images of parallel three-dimensional lines intersect at a point in the retinal plane. This point, which is the image of the common point at infinity of the three-dimensional lines, is called the *vanishing point* of their images. Let us now assume that we have three sets of three-dimensional parallel lines with the further property that their three directions are mutually perpendicular (in practice, those lines could be painted on three faces of a cube). Let m, n, and q be the corresponding vanishing points in the retina.

   (a) Let C be the optical center of the camera. Show that the three angles (Cm, Cn), (Cn, Cq), and (Cq, Cm) are equal to $\frac{\pi}{2}$.

   (b) Let c be the orthocenter of the triangle mnq (i.e., the point of intersection of its three heights). Show that the line Cc is perpendicular to the retinal plane. What conclusion can you draw from this?

   (c) Let us assume that we know the intrinsic parameters of the camera so that we can change coordinate systems in the retinal plane using the matrix $\mathbf{A}$ given in the notes. Show that the matrix $\boldsymbol{\mathcal{P}}$ can be written as

$$\boldsymbol{\mathcal{P}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$$

Find a method for recovering the rotation matrix $\mathbf{R}$ from the three vanishing points m, n, p. Can you generalize it to more than three points?

3. This problem investigates several properties of the image ω of the absolute conic.

   (a) Write down the equation of ω in (u, v) coordinates as a function of the intrinsic parameters and verify that it contains no real points.

   (b) Given two pixels $m_1 = [u_1, v_1, 1]^T$ and $m_2 = [u_2, v_2, 1]^T$, let a and b be the two points of intersection of the line (m_1, m_2) with ω. Compute the cross-ratio of the four points $\{m_1, m_2; a, b\}$.

   (c) From this, compute the angle between the two optical rays L_{m_1} and L_{m_2}.

   (d) Verify your results using standard geometric techniques, i.e. linear algebra.