Geometrical Principles in Motor Control*

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1 Introduction

Geometry is a basic tool for understanding how the central nervous system coordinates the bewildering number of muscles involved even in the simplest gestures. The central role played by geometry in the control of motor behaviors was recognized several decades ago by the Russian investigator Nikolai Bernstein. Using only the tool of logical reasoning applied to common observations, Bernstein formulated a principle of "equal simplicity" (Bernstein 1967). This principle states that the structure of a system determines a set of equivalence classes– the tasks that can be carried out with the same ease. For example, using a compass one can draw circles of different radii without any difference in difficulty. If we can identify these equivalence classes in the observed behavior of a system we can infer something about the system's structure.

Movements such as writing one's signature can be carried out with equal simplicity (and with substantially equal patterns) in different locations of the workspace and in different sizes. This is remarkable because as we write our name on a check or on a blackboard we activate completely different muscles and we move different joints. Nevertheless, our handwriting preserves the characteristic topological features that can be recognized by friends and relatives. "*The hypothesis that there exist in the higher levels of the C.N.S. projections of space, and not projections of joint and muscles* – Bernstein argued – *seems to be at present more probable than any other*". This intuition has led many scientists to consider motor planning and execution as separate processes.

There is a loose but significant correspondence between the separation of planning and execution in motor control and the distinction between kinematics and dynamics in classical mechanics. The term kinematics indicates the analysis of motion in time and space, independent of the forces acting upon the moving system. The analysis of the relationship between forces and motion falls instead within the scope of dynamics. By these definitions, kinematics has become paradigmatic of the abstract processes that we may use to describe a motor plan. Here we adopt this point of view[1] and we begin by reviewing the issue of kinematic planning. Then, we represent the ensuing task of translating planned behaviors into motor commands as a particular type of coordinate transformation. This representation is particularly suitable for addressing some of the most challenging problems that arise from kinematic redundancy – a well known and intriguing feature of biological systems. Finally, we outline a geometrical perspective on the problem of implementing a desired control law from the combination of modules determined a-priori by the peripheral structures of the CNS and by the neuromuscular apparatus.

[1] However one must keep in mind that motor planning is not merely the planning of movement. We may easily define tasks that involve the application of a force upon a fixture or the maintenance of a firm and stable posture against some expected changes in the environment. Clearly, these planned behaviors do not belong to the realm of kinematics.

2 Kinematics

There is a very simple kinematic pattern that may be observed in movements as diverse as the saccade of an eye and a displacement of the upper arm. Namely, when a task is specified as "go from the current location, A, to a new location, B," this pattern consists of a monotonic and temporally–symmetric transition from A to B. Here, "temporally symmetric" refers to a bell–shaped velocity profile: the movement starts from a resting state (zero velocity), reaches peak velocity in a given time and then decelerates back to zero velocity in approximately the same time.

At first, one might be tempted to account for these kinematic features on the basis of the mechanical low–pass filtering properties of bones and muscles. This interpretation, however, does not stand when one considers movements involving the coordination of multiple degrees of freedom. For example, during coordinated eye and head movements to a target, a rapid displacement of the eyeball towards the target is followed midway by a return movement towards the initial location in the orbit (Bizzi *et al.* 1971). Yet, the *gaze*–i.e. the line of fixation in the extrapersonal space–shifted from start to end position in a single monotonic "saccade" which could only be achieved through a precise coordination of head and eye movements. In normal conditions, this coordination is mostly due to the action of the vestibulo–ocular reflex. But, if the vestibular signals are eliminated, then the same coordination is restored by a sensory signal coming from the neck muscles. This demonstrates that the symmetric pattern of movement is likely to reflect some centrally–imposed constraint rather than the peripheral mechanics.

2.1 Multi–joint arm movements

A similar pattern of coordination in a totally different circumstance was reported by Morasso (1981) who asked subjects to move their dominant hand toward a number of visual targets. The subjects tended to move the hand in a straight path from start to end location and the tangential velocity of the hand[2] followed a symmetric bell–shaped profile. However, when Morasso described reaching arm movements in terms of angular displacements of the shoulder and the wrist, he found that the angular motions changed from place to place. In some cases, one or both articulations reversed direction, from flexion to extension or vice versa, revealing a rather complex pattern of intersegmental coordination. These observations are consistent with the idea that movement kinematics reveals the operation of a planning process which is primarily concerned with the generation of simple movement patterns in the geometrical context of the surrounding environment.

[2]If the trajectory of the hand in the plane is described by a pair of cartesian coordinates $x(t)$ and $y(t)$, then the tangential velocity $v(t)$ is given by $v(t) = \left(\dot{x}(t)^2 + \dot{y}(t)^2\right)^{1/2}$. The dot stands for time derivative.

2.2 Smoothness

The simple kinematic pattern described above is the expression of a mathematical property known as *smoothness.* To be more precise, given a sufficiently differentiable function, $f(t)$, representing the position of the observed system vs. time, consider the functional[3]

$$P[f] = \int_{ti}^{tf} \sum_{n=1}^{K} a_n \mid \frac{d^n f}{dt^n} \mid^2 dt \quad (a_K \neq 0). \tag{1}$$

as a measure of the lack of smoothness of $f(t)$ over the interval $[ti, tf]$. The integer, K, determines the highest–order derivative that participates in the smoothness criterion. Given a set of functions, U, one may ask which of these yields a minimum value for P[f]. This is the typical formulation of an optimization principle (Kirk 1970).

Hogan (Hogan 1984) proposed a smoothness criterion, based upon the minimization of the third time derivative of position, also known as "jerk" ($a_1 = a_2 = 0$, $a_3 = 1$). From the Euler equation (Hildebrand 1976) one derives that the optimal function is a polynomial of the fifth degree in time:

$$f(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5.$$

The five constants, $b_0 \ldots b_5$, are determined uniquely from the boundary conditions on f, $\dot{f}$ and $\ddot{f}$ at the extremes of the interval $[ti, tf]$. Hogan found that such a fifth order polynomial is sufficient to reproduce the main kinematic features of one–dimensional motions: a monotonic transition from initial to final position with a symmetric, bell-shaped velocity profile. This result was later challenged by Stein and coworkers (Stein *et al.* 1988) who suggested that a better fit to experimentally observed single–joint reaching motions is provided not by using, for the velocity, the mother of all bell–shaped curves, that is the Gaussian:

$$\dot{f}(t) = C \exp(-(t - t_0)^2 / 2\sigma^2).$$

However, Poggio and Girosi (Poggio and Girosi 1990) showed in a totally different context that Gaussians optimize a smoothness functional which includes all time derivatives up to infinite order – a slightly modified form of $P[f]$ in Equation (1). Therefore, the findings of Stein and coworkers can be taken as a reinforcement rather than a critique of the maximum smoothness principle! Smoothness optimization leads to an interesting dilemma when one considers multi-joint movements. In this case, one must choose to express the functional of Equation (1) either in terms of intrinsic (e.g., joint) or extrinsic (e.g., hand) coordinates. These two choices lead to different solutions, because the mapping between extrinsic and intrinsic coordinates is generally non-linear. Following this argument, Flash and Hogan (Flash and Hogan 1985) demonstrated that the kinematics results of Morasso (1981) can be accounted for by a formulation of minimum jerk in extrinsic hand–coordinates. From this finding they concluded that the motor system is divided into "higher levels which plan ideal

[3]Roughly speaking, a functional is a "function of functions" that is a mapping from a set of functions to the real numbers

trajectories for the end–effector and lower level processes which translate them into torques and forces".

Such a Bernsteinian view has been challenged by Uno et al. (1989) who have proposed a radically different model for explaining multi–joint kinematics on the basis of dynamics optimization. They suggested that the observed trajectories of the arm reflect the minimization of a cost functional in joint coordinates: the time derivative of torque integrated (in the squared norm) along the arm trajectory. Surprisingly, these authors found that their joint–based optimization was able to reproduce with remarkable accuracy the same kinematic patterns captured by jerk optimization in hand coordinates. In addition, their model reproduced better than minimum-jerk the kinematic features of movements between targets located in eccentric regions of the workspace.

However, some recent observations of Shadmehr and Mussa–Ivaldi (1993) and Flash and Gurevich (1992) have provided additional evidence that is not compatible with the notion that movement kinematics emerge from dynamics optimization. Shadmehr and Mussa–Ivaldi asked their subjects to generate reaching movements within a central region of the workspace and in the presence of a velocity–dependent field of forces. Before the application of the field the subjects produced straight-line trajectories with symmetric velocity profiles as described by Morasso (1981), Flash and Hogan (1985) and Uno et al. (1989). Immediately after the presentation of the field, this kinematic pattern was predictably altered by the unexpected and uncompensated forces. However, after a number of movements within the disturbing field subjects showed a considerable and spontaneous adaptation in which the original kinematic patterns– straight hand movements with symmetric velocity profile– were restored. This finding is hardly compatible with dynamic optimizations such as minimum torque–rate because the same smooth endpoint kinematics have been observed under radically different dynamic conditions– before adaptation, with no field, and after adaptation, in the field.

2.3 The unsmooth nature of motor programs

Another controversial issue concerns the relation between smoothness of trajectories and the discontinuous nature of motor "programs". When we write with a pen the movements of our hand must implement a temporal sequence of instructions that corresponds to a sequence of characters. This sequence is inherently discontinuous. In contrast, the motion of the pen is smooth and there is a close temporal correlation between peaks in the curvature and minima in the velocity profile.

Morasso and Mussa–Ivaldi (Morasso and Mussa-Ivaldi 1982) proposed a model of trajectory formation and handwriting based upon the blending of smooth curves, a paradigm equivalent to spline interpolation (Boor 1978). In this model, complex trajectories arise from the superposition of more elementary movements called "strokes". Strokes are blended in time: a new one may start before the current one terminates.

At first, this idea was criticized by Flash and Hogan (1985) who argued that the stroke–combination unlike jerk minimization was not able to "explain" the observed kinematics. However, the already cited study of Poggio and Girosi (1990) has also shown that the two approaches– smoothness optimization and stroke combination– are mathematically equivalent. The solution of a constrained optimization problem– such as finding the curve passing as close as possible to a set of points while minimizing a functional– can be expressed as a weighted combination of predefined basis functions (see the section on radial basis functions in this Handbook). Poggio and Girosi (1990) argued that the same operator which defines the cost functional to be minimized, also defines the family of basis functions that are best suited for expressing the approximating function. In particular, variational smoothness criteria such as minimum–jerk lead to the family of spline functions (Boor 1978) that implement mathematically the stroke–combination mechanism.

The use of discrete strokes allows one to solve the smoothness optimization without deriving the entire optimal trajectory before the beginning of movement execution. The same issue was addressed by Hoff and Arbib (1993) in a recent investigation. They translated jerk minimization into a linear–feedback law that incorporates the hand acceleration in the system state vector. This control law is derived from a family of solutions to the minimum–jerk problem: the family of reaching movements that bring the hand to rest at a given position, starting from a given initial state. The control law proposed by Hoff and Arbib is capable of adjusting a reaching movement after the occurrence of a perturbation in the initial or in the final (planned) position. On the other hand, the stroke–combination mechanism of Morasso and Mussa–Ivaldi is directed to preserve smoothness of a complex trajectory by representing it as a sequence of reaching movements overlapping in time and space. Hence, stroke combination and the minimum–jerk controller may be regarded as cooperating processes that enforce a smoothness criterion over different time scales.

In summary, the analysis of movement kinematics has suggested that the planning and execution of movements are carried out by distinct stages of information processing and that the planning of complex sequential movements is likely to be based upon the superposition of simpler "primitive" motions. These primitive motions can be regarded as elementary units– or "engrams" (Bernstein 1967)– that achieve smoothness through the coordination of all the degrees of freedom involved in any given task.

3 Coordinate systems for motor control

Following up on the distinction between planning and execution of behaviors, one must address the issue of how the central nervous system may transform a planned behavior into action, that is into a pattern of neuromuscular activations. To deal with this issue in mathematical terms, we make use of variables such as position, velocity, acceleration and force. These variables can be measured and represented in a great variety of coordinate systems.

We will not focus on any coordinate system in particular. Instead, we will discuss three general classes of coordinates that correspond to broad categories of description in motor control. These classes are: endpoint coordinates, generalized coordinates and actuator coordinates. Endpoint coordinates provide a representation of the interface between a system and its environment. Therefore, they are the best choice for describing a plan as the desired effect of an action upon the environment. Generalized coordinates support the description of a system's dynamics. Finally, actuator coordinates are used to represent the direct mechanical effect of control signals acting upon the different elements of the system. Each of these classes may be further specified by a particular system of coordinates whose choice depends on the experimental or theoretical context one is dealing with.

The specification of a coordinate system for describing an aspect of motor control is equally relevant for carrying out experimental measurements and for organizing theoretical models of neural information processing. While the investigators of motor behavior and of neural control may adopt different system of coordinates, their choice will necessarily fall in one of these classes.

3.1 Endpoint coordinates

To give some concreteness to our discussion on coordinate systems, consider a monkey in the act of reaching for some food with a wooden stick. The free extremity of the stick, is the site at which the monkey interacts with its environment. Borrowing some terminology from Robotics, we call such a site an "endpoint". The position of any rigid tool is fully determined by a system of six coordinates. These coordinates are measured with respect to three orthogonal axes originating, for example, from the monkey's shoulder. Why six coordinates? Because this is the minimum number of parameters needed to specify unambiguously the location and orientation of a rigid object in 3D space. Then, in our example a position in endpoint coordinates is a point $p = (x, y, z, \theta_x, \theta_y, \theta_z)$. The coordinates, x, y and z determine a translation with respect to the orthogonal axes. The coordinates, θ_1, θ_2 and θ_3, determine an orientation given by three angles with respect to the same axes. Consistent with this notation, a *force* in endpoint coordinates is a vector with three linear and three angular components taken with respect to the same axes:

$$F = (F_x, F_y, F_z, \tau_x, \tau_y, \tau_z)$$

The number of endpoint coordinates varies with the motor task. For example, in Morasso's experiments (Morasso 1981) human subjects kept their arm in the horizontal plane while grasping the handle of a planar manipulandum. The endpoint, in this case, was the location of the center of the handle in the plane and was defined by a pair of cartesian coordinates, $p = (x, y)$.

3.2 Generalized Coordinates

A different way of describing the position of the monkey's arm and of the stick is to provide the set of joint angles that define the orientation of each skeletal segment either in absolute terms (that is with respect to fixed axes in space) or with respect to the neighboring segments. Joint angles are a particular instance of what is known more generally as *generalized coordinates*. According to the standard definitions of analytical mechanics, generalized coordinates are *independent variables* that are suitable for describing the dynamics of a system (Goldstein 1980).

Once we have defined a set of generalized coordinates we may also define a *generalized force* vector in the same geometrical space. For example, if we use joint angles as a system of generalized coordinates, the corresponding components of the generalized force are the torques measured at each joint. The dynamics of any mechanical system with N generalized coordinates are fully described by N second order differential equations relating the generalized coordinate to their first and second time derivatives and to the generalized forces.

In vectorial notation, the dynamics equations for the skeletal system of the monkey's arm can be written as:

$$\tau(q, \dot{q}, u(t)) = I(q)\ddot{q} + C(q, \dot{q}) \tag{2}$$

where $q = (q_1, q_2, \ldots, q_n)$ is the arm configuration in joint–angle coordinates, $\dot{q}$ and $\ddot{q}$ are respectively the first (velocity) and second (acceleration) time derivatives of q, I is a $N \times N$ matrix of inertia (that is configuration–dependent) and $C(q, \dot{q})$ is a vector of centripetal and Coriolis torques (Raibert 1978). The whole right hand side of equation (2) represents the torque due to the inertial properties of the arm. The term $\tau(\cdot)$ on the left–hand side stands for the net torque generated non–linearly by the muscles, by the environment (e.g. the gravitational torque) and by other dissipative elements, such as friction. The variable u is a control parameter– for example a set of neural signals directed to the motoneurons– whose dimension may far exceed the number of generalized coordinates. (For a more detailed discussion about this representation of skeletal dynamics see the article by Shadmehr in this Handbook.)

3.3 Actuator Coordinates

Actuator coordinates offer the most direct representation for the motor output of the central nervous system. A *position* in this coordinate system may be, for example, a collection of muscle lengths, $l = (l_1, l_2, \ldots, l_M)$. Accordingly, a *force* would be a collection of muscle tensions, $f = (f_1, f_2, \ldots, f_M)$. The number, M, of actuator coordinates depends upon how lumped is the model of control under consideration. For instance, one may be dealing with a macroscopic representation of muscles (biceps, triceps etc.) as the elements being controlled ($M \sim 30$). In a detailed model of the neuromuscular system, one should consider each motor unit as an independently controlled element ($M \sim 10,000$). Unlike generalized coordinates, actuator coordinates do not constitute a system of independent variables: we cannot set arbitrary values to l_i without eventually violating some kinematic constraint. Nevertheless, actuators are independent units of control. Indicating with u the state of activation of a muscle, one may say that the central nervous system determines the tension generated by a muscle at any given state $(q, \dot{q})$ of the skeletal system.

4 The workspace and its transformations

Having introduced the three classes of coordinates that support the description of planning, dynamics and control, we now discuss the general issue of transforming mechanical variables among these classes.

Both the transformations from generalized coordinates to endpoint coordinates, and from generalized coordinates to actuator coordinates are, in general, nonlinear mappings. In the example of the monkey's arm, the transformation from joint to hand coordinates is a nonlinear function

$$r = \mathcal{L}(q) \tag{3}$$

where r indicates the position of the arm extremity (the tip of the stick) in endpoint coordinates and q is the joint configuration. The transformation from joint to muscle coordinates is another nonlinear mapping

$$l = \mathcal{M}(q). \tag{4}$$

The above expression reflects the fact that once we know the anatomical parameters of the musculoskeletal apparatus (such as muscles' origins and insertions) we may calculate univocally the muscle lengths from the joint angles.

4.1 The transformation of vectors and vector fields

A function that associates a vector to each point of a multidimensional domain, M, is called a *vector field over M*. For example, we may rearrange the terms of the dynamics equation (2) so as to represent the arm's acceleration, $\ddot{q}$, as a time–varying vector field over the state space described by q and $\dot{q}$:

$$\ddot{q} = I^{-1}(q)(\tau(q, \dot{q}, u(t)) - C(q, \dot{q})). \tag{5}$$

Another vector field describes the viscoelastic behavior of the arm muscles. This behavior can be measured by stimulating each muscle and recording the resulting tension, f, at different muscle lengths, l, rates of shortening, $\dot{l}$, and times, t. Then, the collective mechanical output of the skeletomotor system is summarized by a force field in muscle coordinates

$$f = \alpha(l, \dot{l}, t). \tag{6}$$

Vector fields such as (5) or (6) determine the way in which a system reacts to its environment on one hand and to its control signals on the other. To address the issue of translating a planned action into control we must understand how such mechanical fields are transformed by a change of coordinates.

Let us begin by considering the laws that govern the transformation of a point from a set of coordinates, x, into a new set of coordinates, $\bar{x}$. In this section, we indicate a variable in the new coordinate system by placing a bar on top of the corresponding symbol. The coordinate transformation is a nonlinear function

$$\bar{x} = T(x). \tag{7}$$

We assume that this function is continuous and sufficiently differentiable (the existence and continuity of second partial derivatives is enough for most practical purposes.) However, we do not require the existence of an inverse mapping, $x = T^{-1}(\bar{x})$. In almost all biologically meaningful cases – such as the transformation from joint to muscle coordinates – the two coordinate systems have different dimensions and the inverse mapping doesn't exist.

Next, consider how a vector field, $v(x)$ is related to the corresponding vector field, $\bar{v}(\bar{x})$, in the new coordinate system. First, assume that $v(x)$ is a field of velocity vectors, $\dot{x} = v(x)$, and apply the chain rule:

$$\dot{\bar{x}} = \frac{d\bar{x}}{dt} = \frac{\partial \bar{x}}{\partial x}\frac{dx}{dt} = \frac{\partial \bar{x}}{\partial x}\dot{x}.$$

As we are dealing with multivariate functions, the expression $\partial \bar{x}/\partial x$ represents the functional derivative, or *Jacobian* of the transformation T. This is a position–dependent matrix, $J(x)$, whose elements are

$$J_{i,j}(x) = \frac{\partial \bar{x}_i}{\partial x_j}$$

and the transformation for the velocity field can be rewritten as

$$\dot{\bar{x}} = J(x)\dot{x} \tag{8}$$

A vector that changes according to this law is called *contravariant*[4].

Does Equation (8) provide a rule for transforming the whole velocity field $\dot{x}(x)$ into $\dot{\bar{x}}(\bar{x})$? The answer is affirmative only in the hypothesis – *to which we are not committing ourselves* – that the mapping T is invertible. In fact, in this case we see that

$$\dot{\bar{x}} = J(T^{-1}(\bar{x}))\dot{x}(T^{-1}(\bar{x})) = \bar{v}(\bar{x})$$

In contrast, if T is not invertible we are stuck with the fact that the terms on the right of Equation (8), J and $\dot{x}$, both depend on x and not on $\bar{x}$. Summing up, we know how to transform a contravariant vector *at a given point* but we do not know how to transform a contravariant *field.*

The situation is quite different when dealing with vectors that follow the rule of transformation of the gradient operator. Again, applying the chain rule we have that

$$\frac{\partial}{\partial x} = \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}}.$$

Introducing the Jacobian matrix this expression becomes

$$\frac{\partial}{\partial x} = J^T(x)\frac{\partial}{\partial \bar{x}} \tag{9}$$

A vector following this type of transformation is said to be covariant. Note that the dual or reciprocal nature of (8) and (9). An infinitesimal displacement (or a velocity) is transformed by the Jacobian into the new coordinate system. The same Jacobian maps a covariant vector the other way around– from the new to the old coordinate system.

[4]The notions of contravariance and covariance belong to tensor analysis. A modern–style tutorial on the general principles of this theory can be found in Bishop and Goldberg, (1980).

A typical example of a covariant vector is force, F. The covariance of force can be readily deduced from the tensor invariance of work and power. *Work and power are indeed true scalar variables whose value cannot be modified by a change of coordinates.* In the original coordinate system, power is calculated as

$$F^T \dot{x}$$

and in the new coordinate system, as

$$\bar{F}^T \dot{\bar{x}}.$$

By equating these two expression and introducing (8) we obtain

$$\bar{F}^T J(x) \dot{x} = F^T \dot{x}.$$

Hence, the transformation of an entire field, $\bar{F}(\bar{x})$, is

$$F(x) = J(x)^T \bar{F}(T(x)) \tag{10}$$

This is the same law as (9). Then, we reach the important conclusion that unlike contravariant vectors, *covariant vectors transform globally, as fields.* No inverse transformation is required.

5 Transforming plans into action

We now consider how the discussion of the previous section may shed some light on the problem of implementing a planned behaviors by a pattern of motor commands.

In Section 2 we have reviewed some evidence suggesting that actions are planned in endpoint coordinates. To a certain extent, this can be considered a rather obvious concept that does not require much experimental testing. Indeed, the original statements of Bernstein on this matter were merely based upon circumstantial, everyday–life observations. It is clear for example that we can mentally formulate (and execute) a command such as "trace the shape of a circle with the left hand" without being concerned about the set of muscles that will be involved in this behavior. However, once we have decided to trace a circle we must deal with the problem of choosing which muscles must be activated and in what temporal sequence. In carrying out this task, our brain must face the challenges associated with *kinematic redundancy* – the imbalance between the number of muscle, joint and endpoint coordinates that is typical of any biological system. (See the article of Giszter in this Handbook.)

Before being a source of problems, redundancy is the key feature that enables us to act efficiently within complex environments. Think about the simple act of reaching for the dipstick in a car engine. Not only we can place our hand on target but we can also orient the whole arm so as to avoid touching other parts of the engine and so as to be in a good configuration for exerting a sufficient pulling force. In other words, redundancy gives us flexibility. But, this benefit comes at a computational price: the necessity to deal with ill–posed problems whose solution either is not unique– due to an excess of unknowns– or is not possible– due to an excess of constraints.

The purpose of this section is to show how a proper representation of dynamics and of coordinate transformations leads to a straightforward solution to the problems associated with redundancy. The key for this solution lies in the representation of both the "high level" plans and the "low level" actuator actions as covariant fields of force.

5.1 The transformation of dynamics

The dynamics of a wide variety of mechanical systems are represented by a mapping

$$\ddot{q} = D(q, \dot{q}, t) \tag{11}$$

from the current "state", $(q, \dot{q})$ and time, t, to the acceleration, $\ddot{q}$ (as in Equation 5). Effectively, this is a mapping from current to next state because the acceleration determines the velocity at the next instant of time $(\dot{q} + \ddot{q}dt)$ and the velocity determines the next configuration $(q + \dot{q}dt)$.

The dependence of $\ddot{q}$ upon time in Eq. (11) is primarily meant to represent the action of a control signal upon the dynamic system[5]. For this reason, the presence of such a control signal is often made more explicit by introducing, in the dynamics equation an input function $u(t)$:

$$\ddot{q} = D(q, \dot{q}, u(t))$$

With this representation, one may describe the execution of different tasks through the selection of different input functions.

The above expressions state that a mechanical system's dynamics is a time-varying field of generalized acceleration over the system's state space. Like position and velocity, acceleration can be transformed from generalized to end–point coordinates and from generalized to actuator coordinates but not, in general, the other way around. Hence the discussion on contravariant vector fields of the previous section applies to the particular case of a dynamics field: in a kinematically redundant system, we may map acceleration from generalized coordinates either to endpoint or to actuator coordinates. However, in a kinematically redundant system there is not a well-posed rule for mapping an acceleration *field* to either endpoint or actuator coordinates. Essentially, we have just stated in another way that dynamics can only be properly represented in a system of generalized coordinates: *one cannot achieve a consistent description of dynamics in terms of muscle states and muscle forces or in terms of endpoint states and endpoint forces.* Then, how can we represent dynamic plans and translate them into actions?

This problem is addressed and solved by specifying motor plans and actuator actions as force fields in the respective coordinate systems. First, we must slightly modify the representation of dynamics (11) and replace it with a mapping

$$\ddot{q} = D(q, \dot{q}, \Phi(q, \dot{q}, t)) \tag{12}$$

[5]Other time–dependencies are given by the changes in the mechanical structure of the systems and by the changes in the environmental dynamics. The present framework can easily be adapted to include the modeling of these changes.

from the current state *and a time-dependent field of generalized forces*, Φ, defined over the system's state space, to the acceleration, $\ddot{q}$. All what we have done is to represent the input to the system as a time–varying field of generalized forces. This is a quite general formulation that includes, as a particular case, the torque function τ of Equation (2). Note that the field Φ may depend upon state and time. When it depends only upon time, one is dealing with a feedforward "forcing function". When Φ depends continuously both upon state and time, the control system acts as a combination of nonlinear springs and dampers whose parameters change in time.

Next, consider that a variety of motor plans may be expressed as fields of force in endpoint coordinates, that is as

$$F = \pi(x, \dot{x}, t). \tag{13}$$

For example, a reaching movement of the hand may be planned, as proposed by Flash, by specifying a time–varying field whose equilibrium point moves along a smooth trajectory (see Shadmer's article in this Handbook). In contrast, a constant and uniform force field, $F = F_0$ encodes the task of exerting a force of given direction and amplitude, regardless of the uncertainty with which the hand location may be determined. Following this approach, Kathib (1986) proposed to plan the movements of a robotic arm in an obstacle–ridden environment by associating a field of repulsive forces to each obstacle and a field of attractive forces to the target location . Note that, by representing a plan as a force field one does not specify the dynamics but rather a constraint that eventually the dynamics must satisfy. For example a field converging to a point will eventually lead to a resting state at that point.

From the considerations in section 4.1 one may conclude that the planned field, $\pi(x, \dot{x}, t)$, has a unique image, $\Phi_\pi(q, \dot{q}, t)$ in configuration space. This is true regardless of the degree of kinematic redundancy. The only condition for Φ_π to be defined is that the kinematic transformation from generalized to endpoint coordinates is defined with its first partial derivatives. Operationally, we may construct Φ_π as a combination of three mappings:

$$(q, \dot{q}, t) \xrightarrow{l_1} (x, \dot{x}, t) \xrightarrow{\pi} F \xrightarrow{l_2} Q. \tag{14}$$

The first mapping, l_1, is given by the direct kinematics transformation ($q \rightarrow x$), its Jacobian ($q, \dot{q} \rightarrow \dot{x}$) and the identity function ($t \rightarrow t$). The second mapping is the postulated endpoint field, $\pi(x, \dot{x}, t)$. Finally, the third mapping, l_2, is the transformation from end-point to generalized force which again is provided by the Jacobian of the direct kinematics. Once we have expressed the planning field in generalized coordinates, we can readily derive the system dynamics

$$\ddot{q} = D(q, \dot{q}, \Phi_\pi(q, \dot{q}, t))$$

On the other end of the planning/execution problem, one must deal with a number of actuators that, in any biological system far exceed the number of generalized coordinates. As in Equation (6), we represent the mechanical action of the actuators under a particular pattern of control as a force field

$$f = \alpha(l, \dot{l}, t). \tag{15}$$

As we assume that there is a unique mapping (Equation 4) from generalized to muscle coordinates, we are again in a position to state that the field (15) has a well defined image in generalized coordinates, $\Phi_\alpha(q, \dot{q}, t)$. Again, this field can be derived from the combination of three direct transformations, namely:

$$(q, \dot{q}, t) \stackrel{m_1}{\rightarrow} (l, \dot{l}, t) \stackrel{\alpha}{\rightarrow} f \stackrel{m_2}{\rightarrow} Q. \tag{16}$$

The first transformation, m_1, is the actuator kinematics ($q \rightarrow l$) together with its Jacobian ($q, \dot{q} \rightarrow \dot{l}$) and the identity mapping ($t \rightarrow t$). The second transformation is the actuator force field, $\alpha(l, \dot{l}, t)$, and the third transformation, m_2, is again given by the Jacobian of $\mathcal{M}$.

The generalized force field, Φ_α, induces the dynamics

$$\ddot{q} = D(q, \dot{q}, \Phi_\alpha(q, \dot{q}, t)).$$

We may regard this dynamics as the description of a "pattern generator" corresponding to a particular setting of the control signals. On the other hand, the field Φ_π and its associated dynamics correspond to a planned behavior. Clearly, if $\Phi_\pi = \Phi_\alpha$ this planned behavior is exactly implemented by the given pattern of actuators. However, such a fortunate situation may not be attainable and one must be content with an approximation of the planned goal. The argument that we have illustrated so far indicates that such an approximation can be properly represented in a system of generalized coordinates, regardless of kinematics redundancy. For example, in some cases we may describe the the action of the low–level controllers by means of K independent parameters, $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then, the generalized force field corresponding to the action of the actuators may be written as $\Phi(q, \dot{q}, t \mid \alpha_1, \alpha_2, \ldots, \alpha_K)$ and the approximation task becomes equivalent to that of finding a set of parameters that optimize the error

$$\| \Phi_\pi(q, \dot{q}, t) - \Phi(q, \dot{q}, t, \mid \alpha_1, \alpha_2, \ldots, \alpha_K) \|$$

for a suitable definition of the "field norm", $\| \cdot \|$.

6 Vector fields and the geometry of control

A question that has frequently haunted motor physiologists is "what is coded by motor commands"? On the basis of the correlation between neural activities and mechanical variables, different targets for neural coding have been proposed such as force (Evarts 1968), final position (Polit and Bizzi 1979) and movement direction in extrapersonal space (Georgopoulos *et al.* 1982). Why does motor coding appear to be so elusive? According to Bernstein (1967), "*movements are possible under conditions of the most accurate and uninterrupted agreement– unforeseen in advance– between the central impulses and the events occurring at the periphery, and are frequently quantitatively less dependent on these central impulses that on the external force field.*

Electrophysiological experiments on the spinalized frog have indicated that the activation of a small area within the gray matter of the lumbar spinal cord induces the recruitment of multiple leg muscles (see the article by Giszter in this Handbook). The muscular activations generate a time–varying field of forces acting on the frog's ipsilateral ankle. This field was found to be characterized at each instant of time by a pattern of forces converging towards a location in the ankle's workspace. These results suggested that the spinal interneurons are organized in a small number of functional modules. Each module determines the recruitment of a group of muscles. As a consequence of muscle elastic properties, the activation of a module, i, leads to the generation of a force field, $\phi_i(q,t)$ that converges toward a stable posture of the leg.

How can the central nervous system generate a significant repertoire of behaviors from the activation of a few modules? An answer to this question came from the empirical observation that the simultaneous stimulation of two spinal sites results in the vectorial sum of the fields obtained from the isolated stimulation of each site. This finding suggests that the combination of k independent control modules generates a net field

$$\Phi(q,t,\alpha_1,\ldots,\alpha_k) = \sum_{i=1}^{k} \alpha_i \phi_i(x,t) \tag{17}$$

where α_i are activation coefficients.

This approach was explored in a theoretical investigation by Mussa–Ivaldi and Giszter (1992) who concluded that a wide repertoire of field features can indeed be accurately reproduced by adding only a few convergent fields as in Equation (17). Following the paradigm outlined in the previous section, a control system may approximate a a "target field" $\Phi_\pi(q,t)$ by determining a set of parameters, α_i^*, such that the norm $\| \Phi(q,t,\alpha_1^*,\ldots,\alpha_k^*) - \Phi_\pi(q,t) \|$ is at a minimum. This least–squares method requires the definition of an inner–product structure for the linear space of vector fields. Given two fields $f(x,t)$ and $g(x,t)$ one may define an inner–product of f and g as

$$< f,g >= \int\int f^T(x,t)\cdot g(x,t)p(x,t)dxdt$$

where $p(x,t)$ is a weighting function. Then, the norm of a field, $\| f \|$ is

$$\| f \| = < f,f >^{1/2} .$$

The norm of the error field is minimized by solving for α_i the K equations

$$< \phi_j, \Phi_\pi >= \sum_i < \phi_j, \phi_i > \alpha_i \quad (j = 1,\ldots,K).$$

In essence, each coefficient is obtained by taking the projections, $< \phi_j, \Phi_\pi >$, of the desired vector field along the system of fields generated by the control modules.

In this geometrical framework the execution of an arbitrary task can be regarded as a form of "negotiation" between planning and control processes. On one hand the planning processes may require a field on the basis of the features of the task and without too much of a concern for the properties of the hardware. On the other hand the control processes may simply try to execute the best possible approximation to the planned field. As the quality of this approximation depends upon the structure of the fields, ϕ_i, the adaptation of the control modules to the statistics of required behaviors is likely to play a crucial role in motor learning.

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