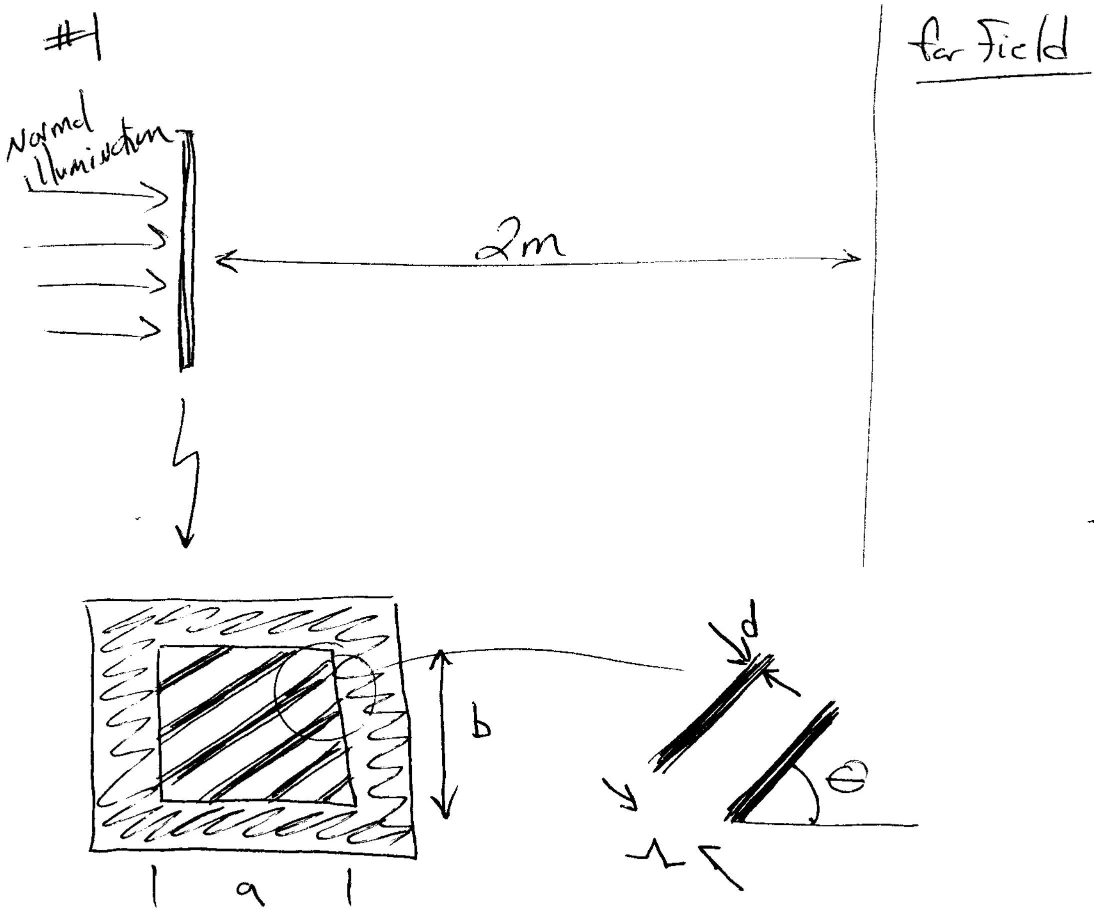
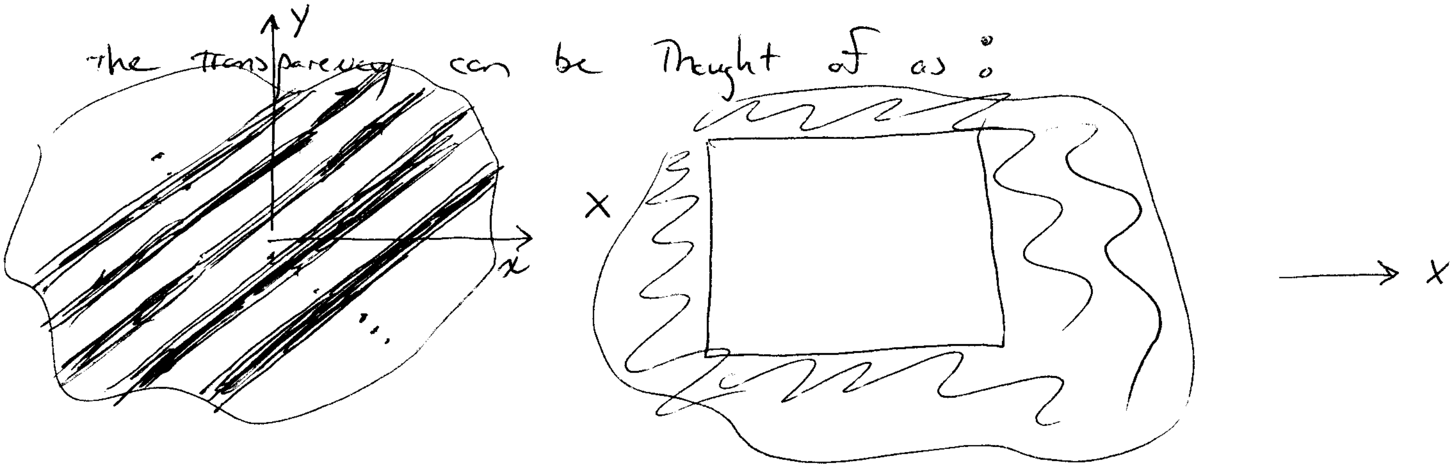
Problem set #4
Fall 2000
2.717
Solutions

#1

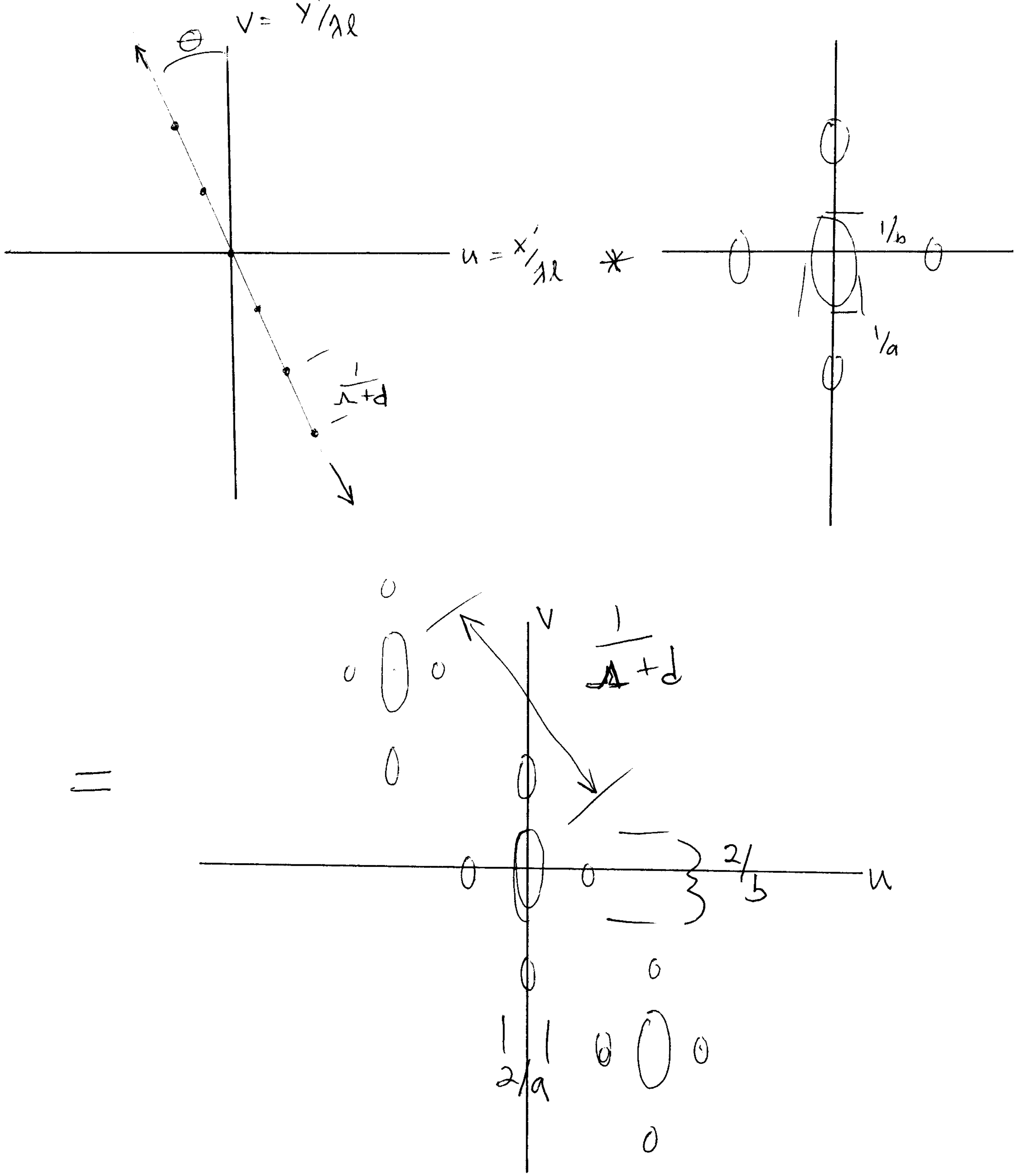


$\Lambda = 10\,\mu m$
$d = 2\,\mu m$
$\Theta = 30^\circ$
$a \times b = 5 \times 3\ mm^2$
$l = 2m$
$\lambda = .5\,\mu m$

The transparency can be thought of as:

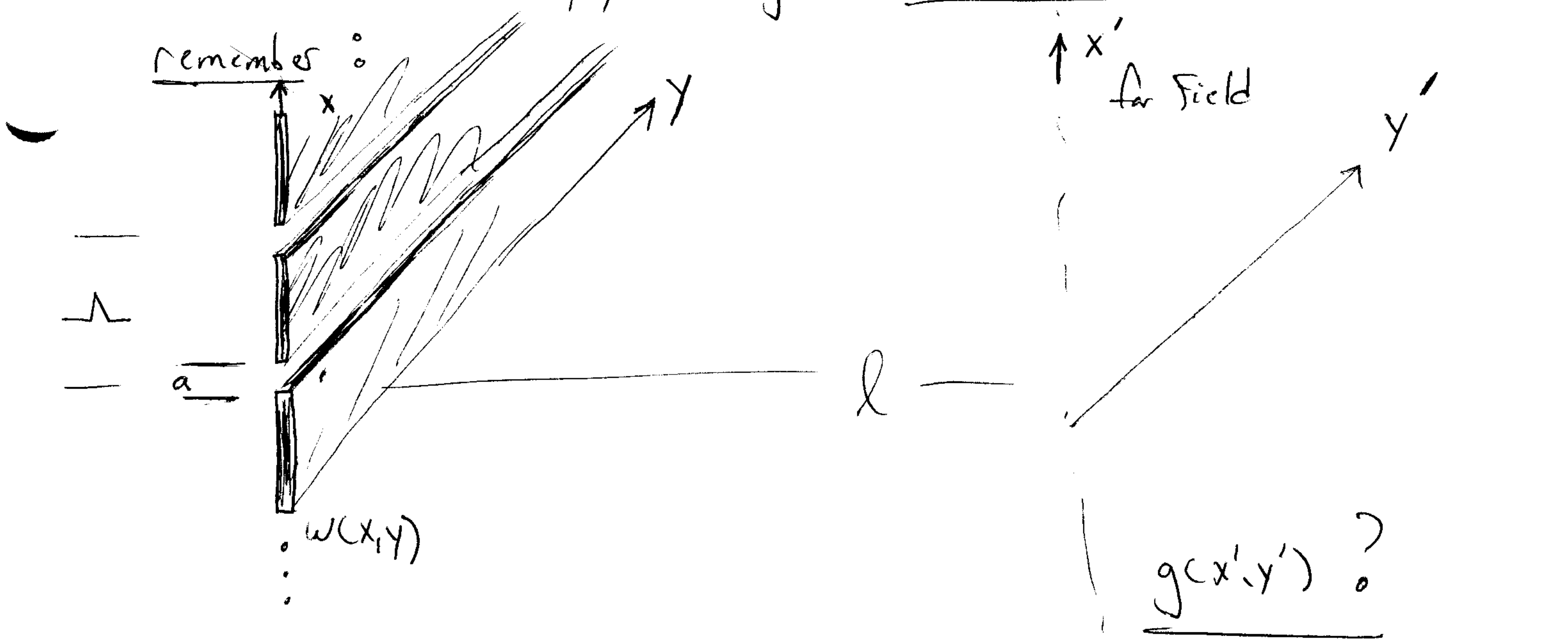


i.e. multiplication in the spatial domain.
$\Leftrightarrow$ convolution in the freq domain. (Far field result)



for details see following pages.

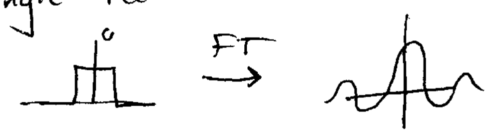
Details: Note: This is much more math than you had to do. Simply drawing the end result was sufficient.



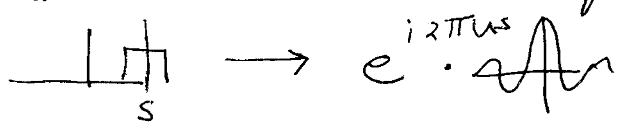
$$g(x',y')_{\text{far Field}} = \mathcal{FT}\left(w(x,y)\right)\Big|_{\substack{u = x'/\lambda l \\ v = y'/\lambda l}}$$

$$W(u,v) = \int \underbrace{\sum_{j=1}^{N} \text{rect}\left(\frac{x-(j-1)\Lambda}{a}\right)}_{= w(x,y)} e^{-i2\pi u x}\, dx$$

Note: • a single rect's FT is the sinc



• a shifted function in the spatial domain is a modulation in freq



so we have a bunch of ~~shifted~~ modulated, summed sincs

$$W(u,v) = \sum_{j=1}^{N} a\,\text{sinc}(au)\, e^{i2\pi(j-1)\Lambda u}$$

$$g(x',y') = \sum_{j=1}^{N} a\,\text{sinc}\left(a\frac{x'}{\lambda l}\right) e^{-i2\pi(j-1)\Lambda\, x'/\lambda l}$$

$$g(x',y') = a\,\mathrm{sinc}\left(a\frac{x'}{\lambda l}\right)\sum_{j=1}^{N} e^{-i2\pi(j-1)\Lambda x'/\lambda l}$$

$$= a\,\mathrm{sinc}\left(a\frac{x'}{\lambda l}\right)\sum_{j=1}^{N} e^{-i(j-1)\gamma} \qquad \gamma = \frac{2\pi\Lambda x'}{\lambda l}$$

$$= a\,\mathrm{sinc}\left(a\frac{x'}{\lambda l}\right)\sum_{j=0}^{N-1} \left(e^{-i\gamma}\right)^{j}$$

Note:

$$\sum_{j=0}^{N-1}\left(e^{-i\gamma}\right)^{j} = \frac{1-\left(e^{-i\gamma}\right)^{N}}{1-e^{-i\gamma}}$$

$$= \frac{e^{-i\alpha N/2}\left(e^{i\alpha\frac{N}{2}} - e^{-i\alpha\frac{N}{2}}\right)}{e^{-i\alpha/2}\left(e^{i\alpha/2} - e^{-i\alpha/2}\right)}$$

$$= e^{-i\alpha(N-1)/2}\,\frac{\sin\left(\gamma\frac{N}{2}\right)/2i}{\sin\left(\gamma/2\right)/2i}$$

$$= e^{-i\pi(N-1)\frac{\Lambda x'}{\lambda l}}\,\frac{\sin\left(\frac{\pi N\Lambda x'}{\lambda l}\right)}{\sin\left(\frac{\pi\Lambda x'}{\lambda l}\right)}$$

so

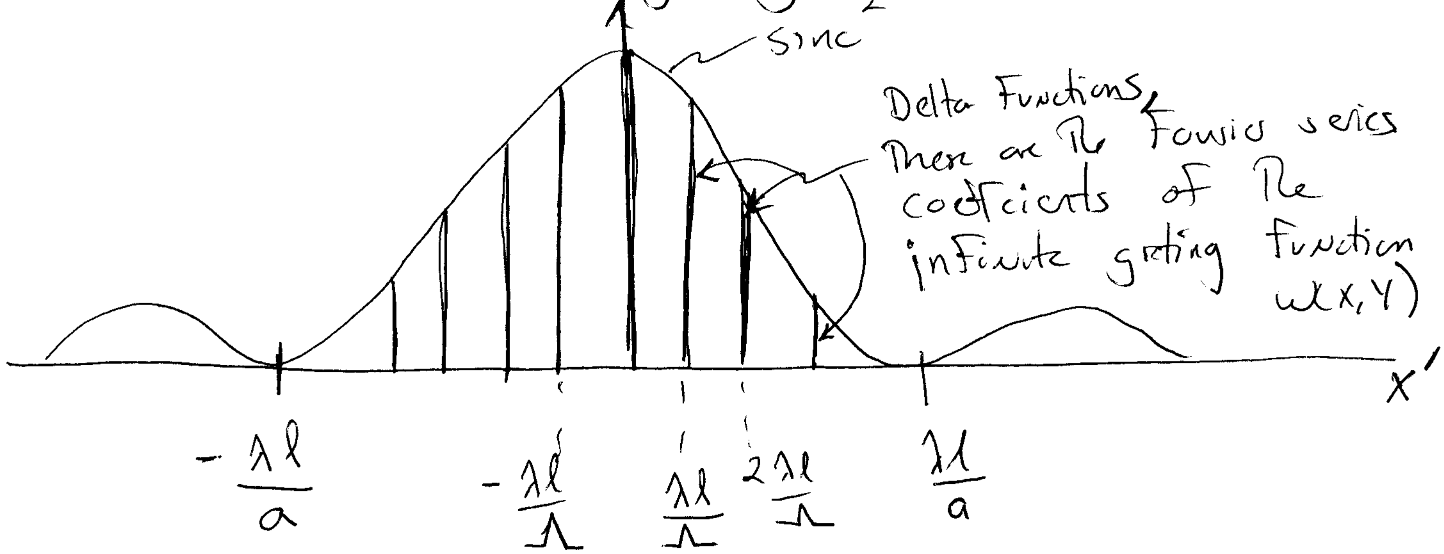
$$g(x',y') = a\,\mathrm{sinc}\left(a\frac{x'}{\lambda l}\right) e^{-i\pi(N-1)\Lambda x'/\lambda l}\,\frac{\sin\left(\frac{\pi N\Lambda x'}{\lambda l}\right)}{\sin\left(\frac{\pi\Lambda x'}{\lambda l}\right)}$$

we observe intensity

$$|g(x',y')|^2 = a^2 \,\mathrm{sinc}^2\!\left(\frac{a x'}{\lambda l}\right) \boxed{\frac{\sin^2\left(\frac{\pi N \Lambda x'}{\lambda l}\right)}{\sin^2\left(\frac{\pi \Lambda x'}{\lambda l}\right)}}$$

when $N \to \infty$ This ↗ goes to a <u>delta function</u>

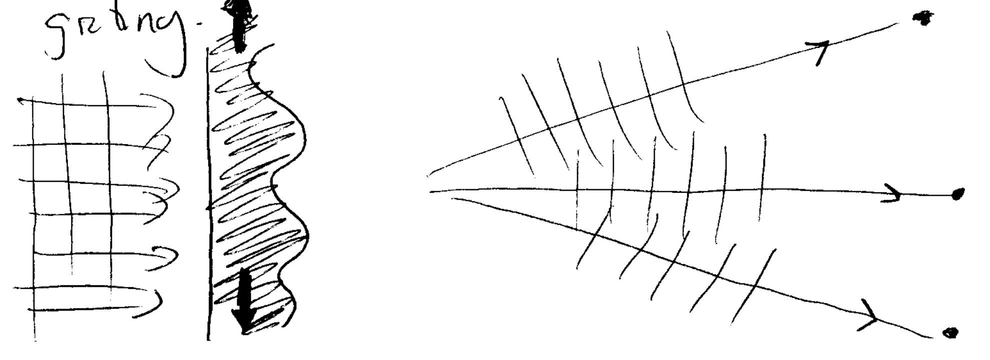
so for an infinite grating:



As an aside: the F.T. of  is $\delta(f) + \delta(f-w_0) + \delta(f+w_0)$

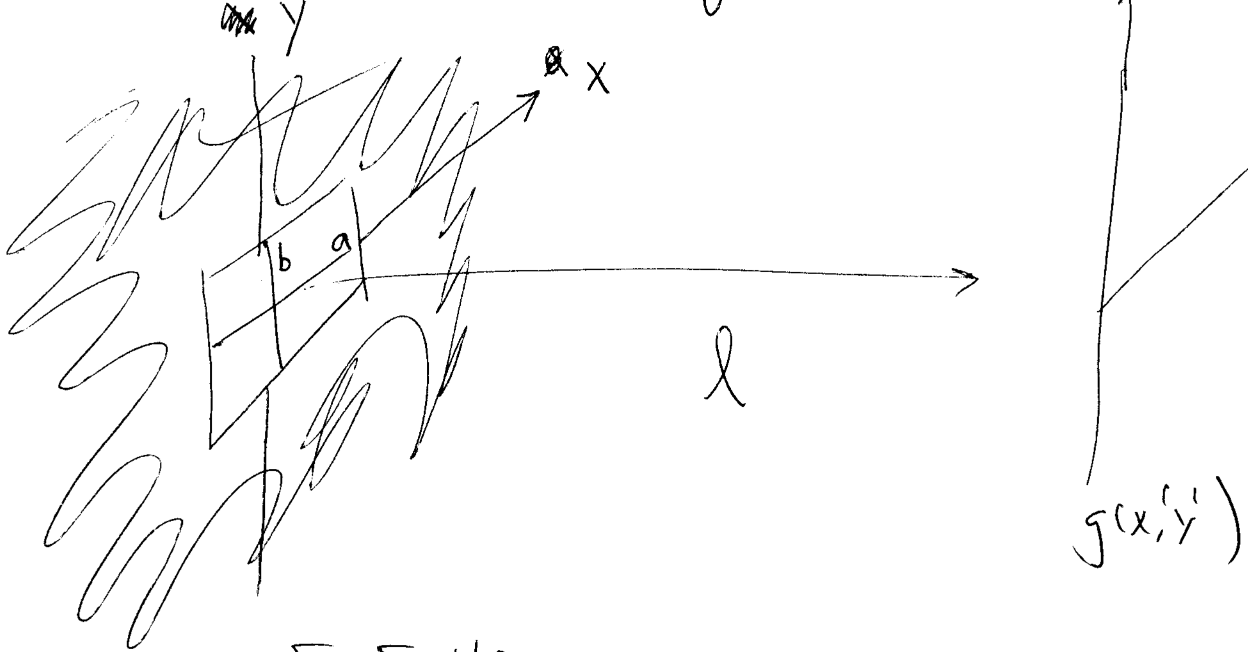
which is what we saw for an infinite sinusoid!

grating:



so this is a general result for infinite gratings.

similarly for a square aperture:



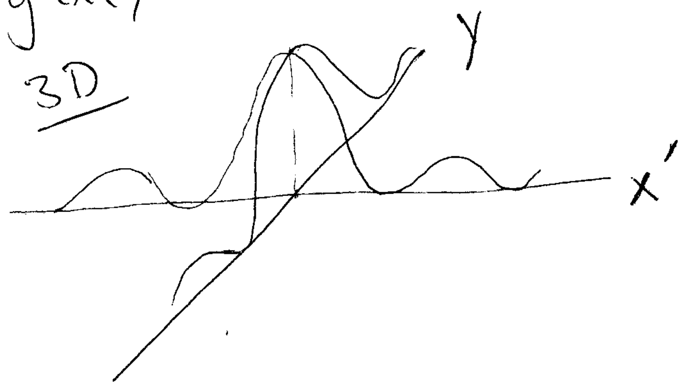
in far field:

$$g(x',y') = \text{F.T.}\big(w(x,y)\big)\Big|_{\substack{u = x'/\lambda l \\ v = y'/\lambda l}}$$

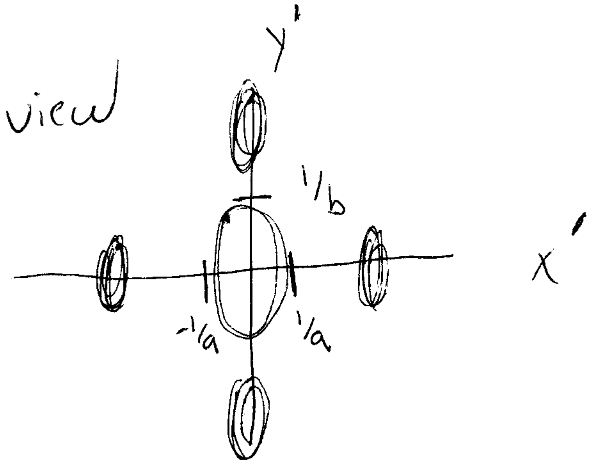
$$= ab\,\mathrm{sinc}\left(a\,x'/\lambda l\right)\mathrm{sinc}\left(b\,y'/\lambda l\right)$$

$g^2(x',y')$

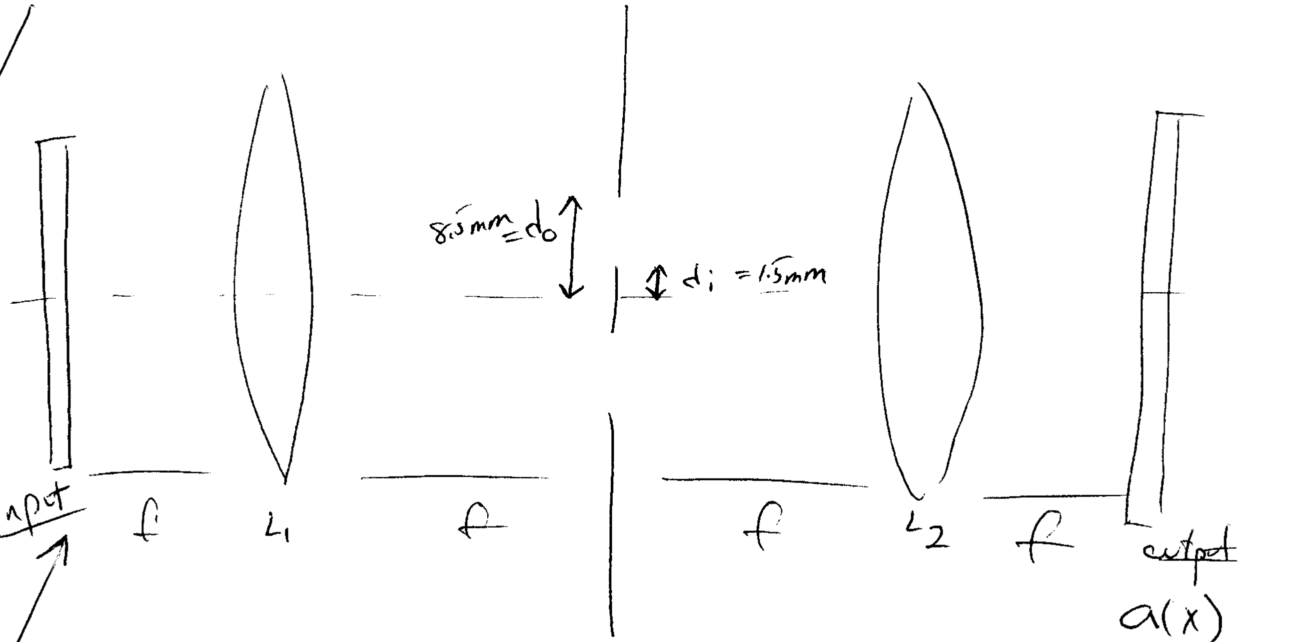
3D

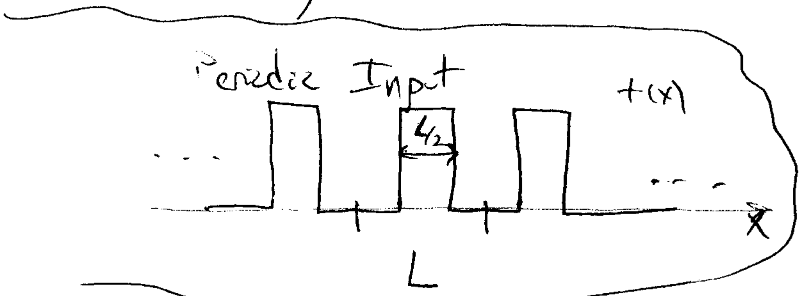


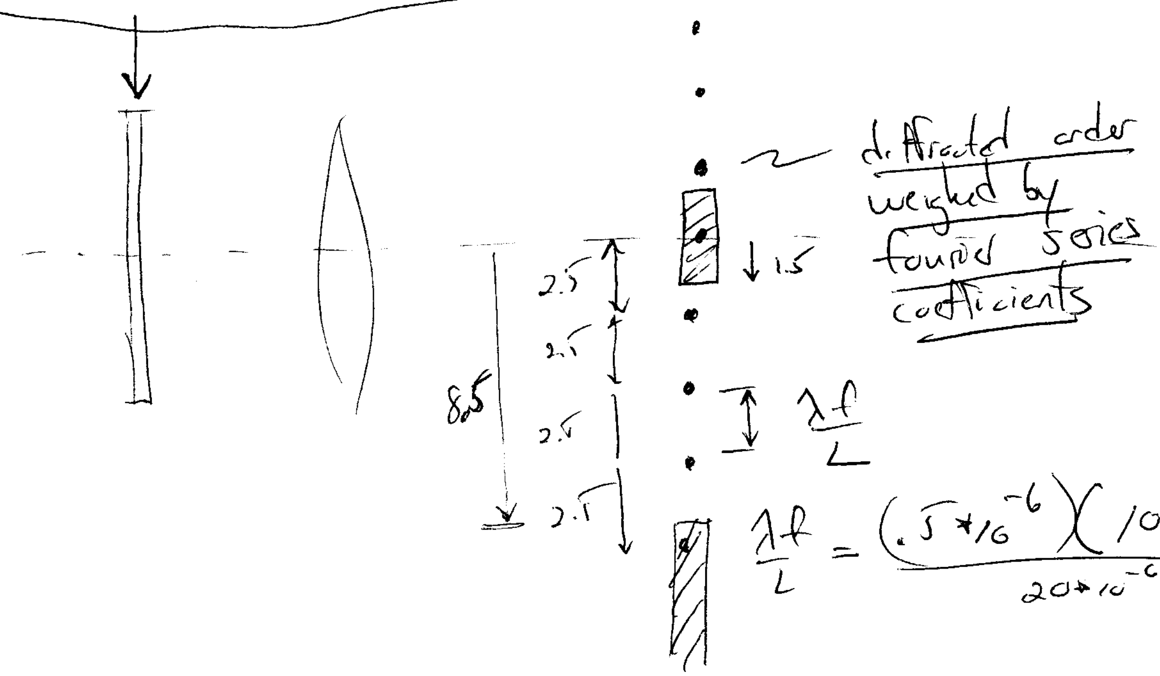
top view



#2







$$\frac{\lambda f}{L} = \frac{(.5 \times 10^{-6})(10 \times 10^{-2})}{20 \times 10^{-6}} = 2.5\ mm$$

mask allows order ±1, ±2, ±3 to pass.

so intuitively in the output plane we have elliminated <u>DC</u> & <u>High Freq Terms</u>

$$c_1 = 1/\pi = c_{-1}$$

$$c_2 = 0 = c_{-2}$$

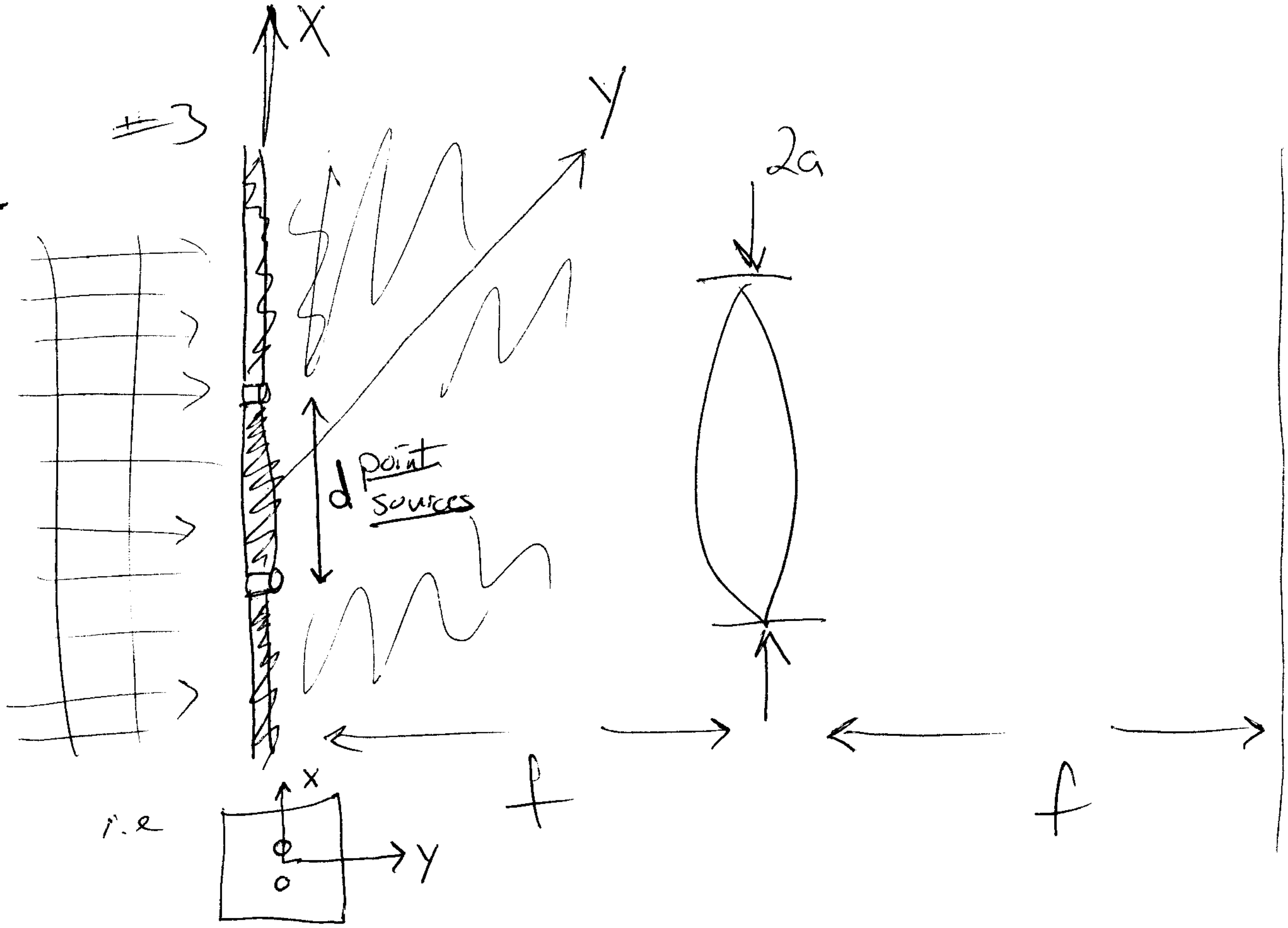
$$c_3 = -1/3\pi = c_{-3}$$

$$a(x) = \frac{1}{\pi}\left(e^{-i2\pi\frac{x'}{L}} + e^{i2\pi\frac{x'}{L}}\right)$$

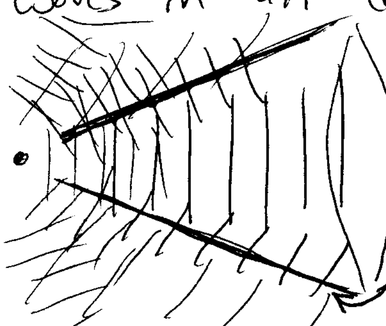
$$- \frac{1}{3\pi}\left(e^{-i2\pi\frac{3x'}{L}} + e^{i2\pi\frac{3x'}{L}}\right)$$

$$a(x) = \frac{2}{\pi}\left[\cos\left(2\pi\frac{x'}{L}\right) - \frac{1}{3}\cos\left(2\pi\frac{3x'}{L}\right)\right]$$

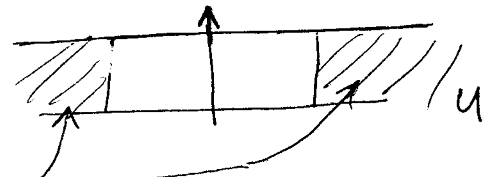
#3



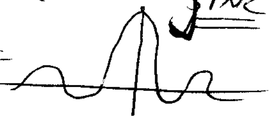
There are several equivalent ways to look at This
First imagine a single point source, it radiates plane waves in all directions (i.e The FT of a point is all freq)



The lens cuts off a freq (angles) past This line

FT of (•) =  u

since circular

The aperture cuts off some of These high frequencies
at The image plane we get The Fourier Transform of what was passed i.e FT of  =  jinc

we saw This in The last homework for a circular aperture

Another way is to say that for an imaging system the output field is the input field convolved with the lens / image system impulse response

i.e

$$g(x',y') = g(x,y) * h(x,y)$$

so for a single point & $\underbrace{\text{1:1 image system}}_{\text{(circular lens)}}$

$$g(x',y') = \delta(x,y) * \frac{2J(\pi a \rho/\lambda f)}{\pi a \rho/\lambda f} \quad \text{where } \rho = (x^2+y^2)^{1/2}$$

$$= \frac{2J(\pi a \rho/\lambda f)}{\pi a \rho/\lambda f} \leftarrow \text{"jinc"}$$

since only interested along x' axis $y'=0$

$$= \frac{2J(\pi a x/\lambda f)}{\pi a x/\lambda f}$$

Now for two point sources:

$$g(x',y') = \left[\delta(x-d/2) + \delta(x+d/2)\right] * \frac{2J(\pi a x/\lambda f)}{\pi(ax/\lambda f)}$$

$$g(x',y') = \frac{2J\left(\frac{\pi a}{\lambda f}(x-d/2)\right)}{\frac{\pi a}{\lambda f}(x-d/2)} + \frac{2J\left(\frac{\pi a}{\lambda f}(x+d/2)\right)}{\frac{\pi a}{\lambda f}(x-d/2)}$$

- The previous was for a coherent source

$$\text{Intensity}(x,y) = |\text{field}(x,y)|^2$$

- for incoherent

$$\text{Intensity}(x,y) = |\text{"field" in}|^2 * \left|\begin{smallmatrix}\text{system}\\ \text{impulse}\end{smallmatrix}\right|^2$$

so

$$\underset{\text{coherent}}{I(x',y')} = \left[\frac{2J\left(\frac{\pi a}{\lambda f}(x'-d/2)\right)}{\frac{\pi a}{\lambda f}(x'-d/2)} + \frac{2J\left(\frac{\pi a}{\lambda f}(x'+d/2)\right)}{\frac{\pi a}{\lambda f}(x'+d/2)}\right]^2$$

we are interested in

$$d = 3.55\,\frac{\lambda f}{a} \qquad 1.22\,\frac{\lambda f}{a} \qquad .41\,\frac{\lambda f}{a}$$

so in normalized coordinates

$$\underset{\text{coherent}}{I(x',y')} = \left[\frac{2J\left(\pi(\bar{x}-\bar{d}/2)\right)}{\bar{x}-\bar{d}/2} + \frac{2J\left(\pi(\bar{x}+\bar{d}/2)\right)}{\bar{x}+\bar{d}/2}\right]^2$$

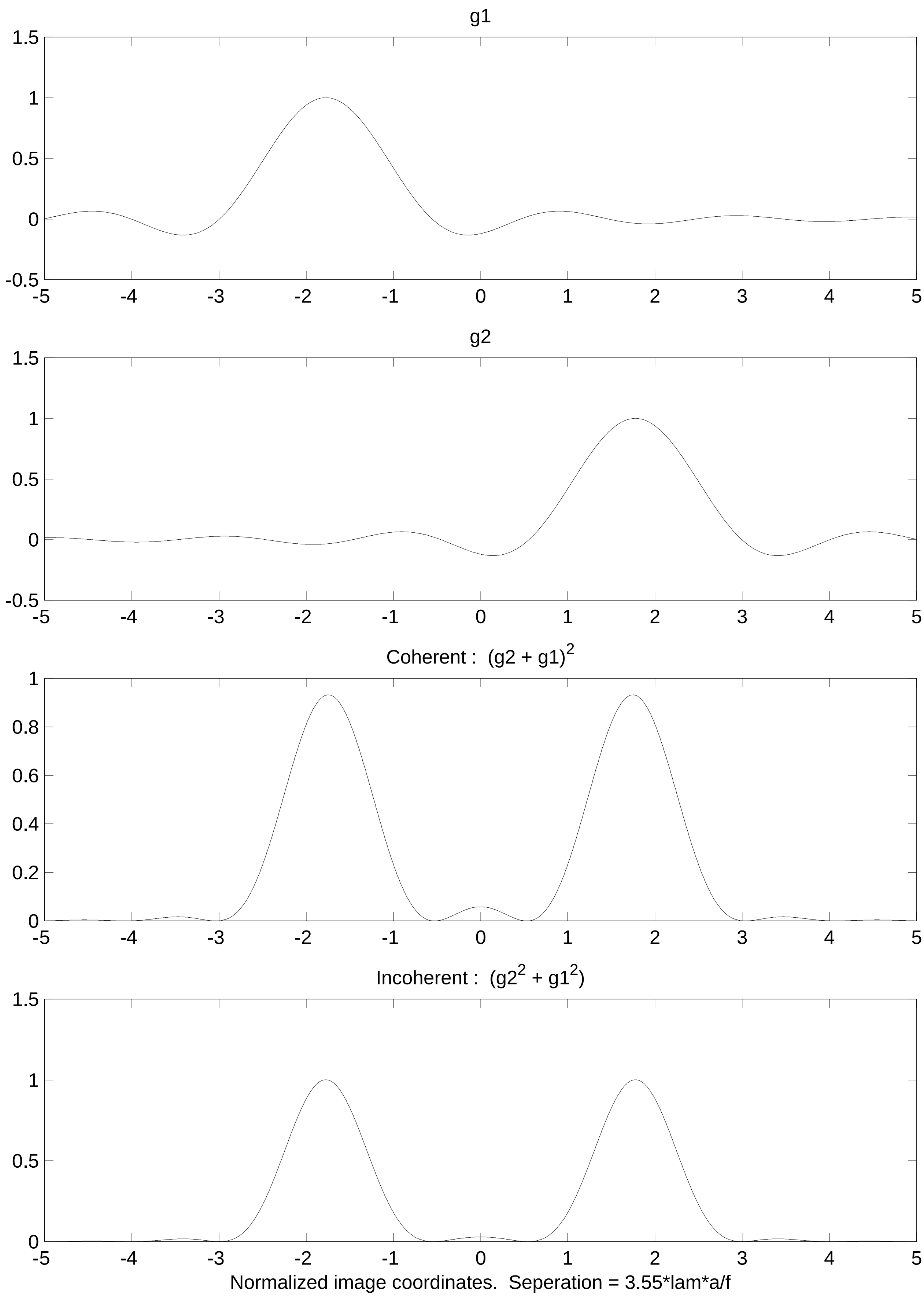
w/ $\bar{d} = 3.55,\ 1.22,\ 0.41$

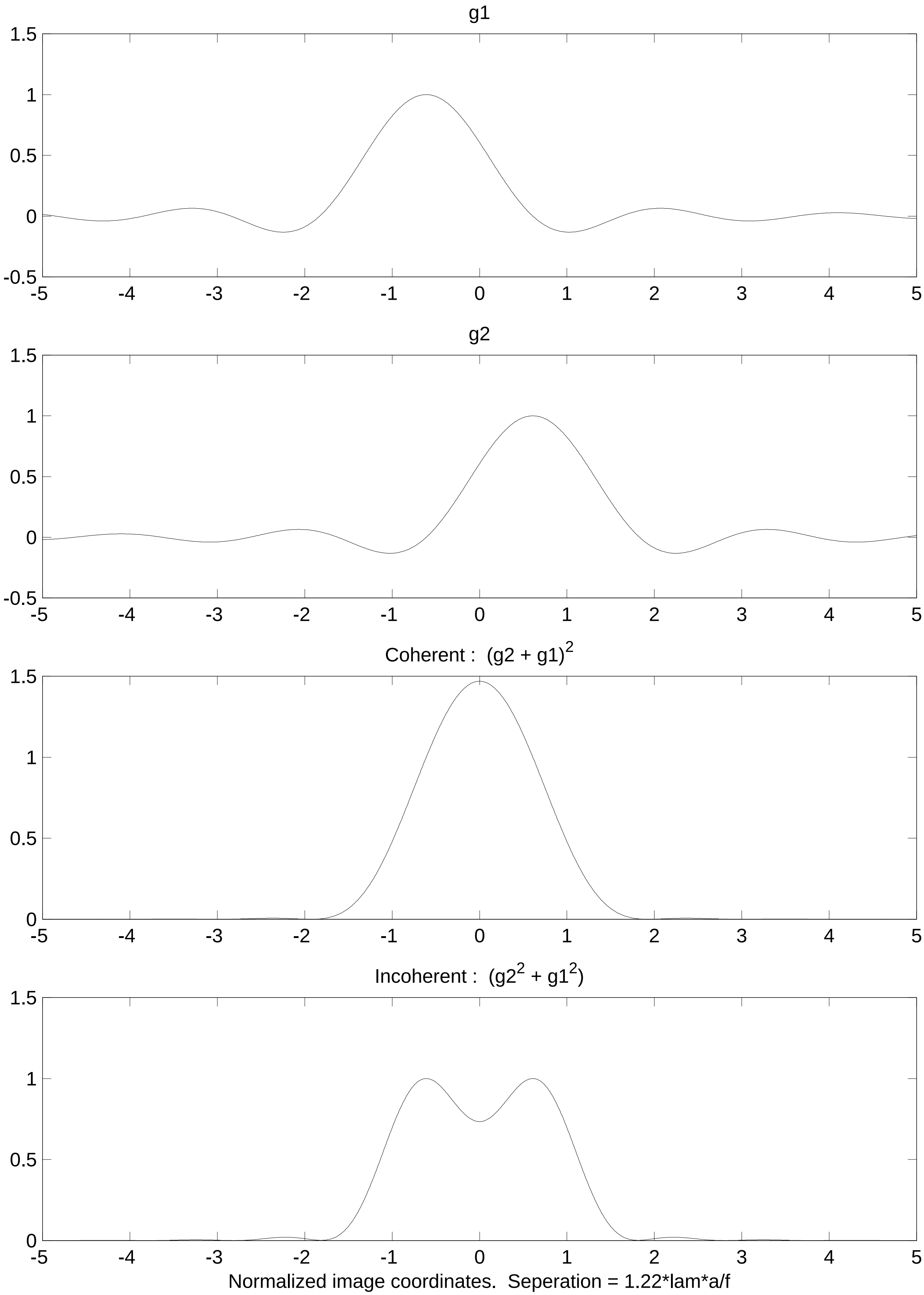
and for incoherent:

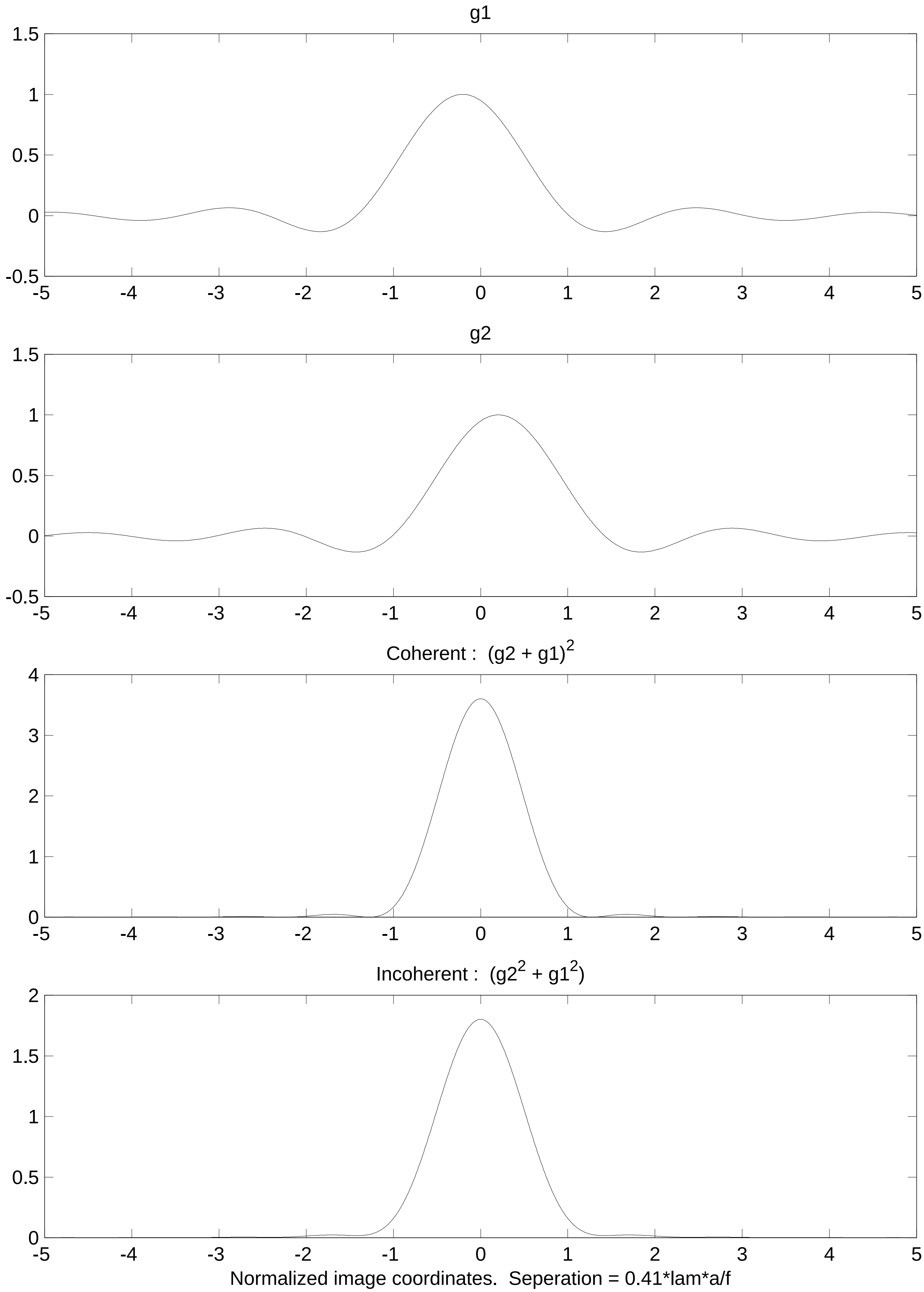
$$\underset{\text{incoherent}}{I(x'y')} = \left[\frac{2J\left(\pi(\bar{x}-\bar{d}/2)\right)}{\bar{x}-\bar{d}/2}\right]^2 + \left[\frac{2J\left(\pi(\bar{x}+d/2)\right)}{\bar{x}+d/2}\right]^2$$

see following plots.

```
%problem set #4
%problem # 3
%2 point resolution
%Brian Anthony

x = -5:.01:5;

sep = 3.55
%sep = 1.22
%sep = .41

g1 = 2*besselj(1,pi*(x+sep/2))./(pi*(x+sep/2));
g2 = 2*besselj(1,pi*(x-sep/2))./(pi*(x-sep/2));
goutC = (g1 + g2).^2;
goutI = (g1.^2 + g2.^2);

figure(1)
subplot 411
plot(x,g1)
title('g1')

subplot 412
plot(x,g2)
title('g2')

subplot 413
plot(x,goutC)
title('Coherent :  (g2 + g1)^2')
zoom on

subplot 414
plot(x,goutI)
title('Incoherent :  (g2^2 + g1^2)')
zoom on


str = ['Normalized image coordinates.  Seperation = ' num2str(sep) '*lam*a/f'];


xlabel(str)
```


g1
g2
Coherent : (g2 + g1)^2
Incoherent : (g2^2 + g1^2)
Normalized image coordinates. Seperation = 3.55*lam*a/f

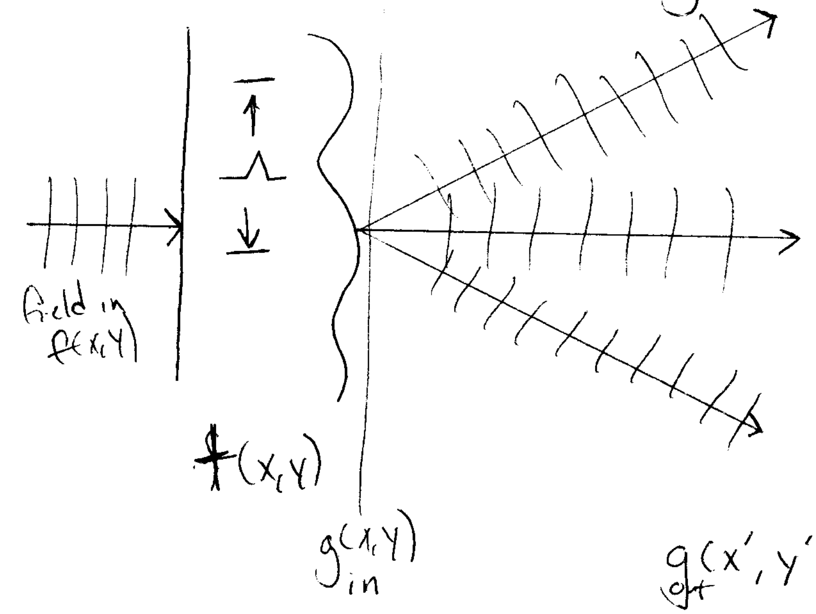

g1
g2
Coherent : $(g2 + g1)^2$
Incoherent : $(g2^2 + g1^2)$
1.5
1
0.5
0
-0.5
-5
-4
-3
-2
-1
0
1
2
3
4
5
Normalized image coordinates. Seperation = 1.22*lam*a/f


g1
g2
Coherent : (g2 + g1)^2
Incoherent : (g2^2 + g1^2)
Normalized image coordinates. Seperation = 0.41*lam*a/f

#4

First,

what does a sine grating do??



3 diffracted orders.

at Far Field $(\to\infty)$

$$g_{out}(x',y') = FT\big(\underbrace{t(x,y) \times f_{in}(x,y)}_{g_{in}}\big)$$

if • f_{in} is a plane wave $= 1$

• $t(x,y)$ sine grating

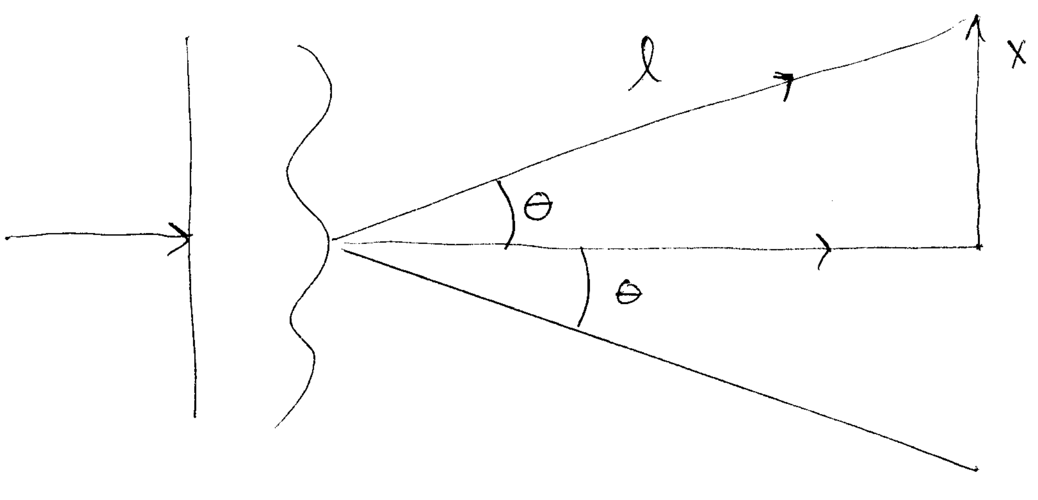
$$= \delta(u,v) + \delta\left(u - \tfrac{1}{\Lambda}\right) + \delta\left(u + \tfrac{1}{\Lambda}\right)$$

where $u = x'/\lambda l$

The point at infinity is at what angle?

well it occurs where $u = \frac{1}{\Lambda}$ or $\frac{x'}{\lambda l} = \frac{1}{\Lambda}$

$$\frac{x'}{l} = \frac{\lambda}{\Lambda}$$

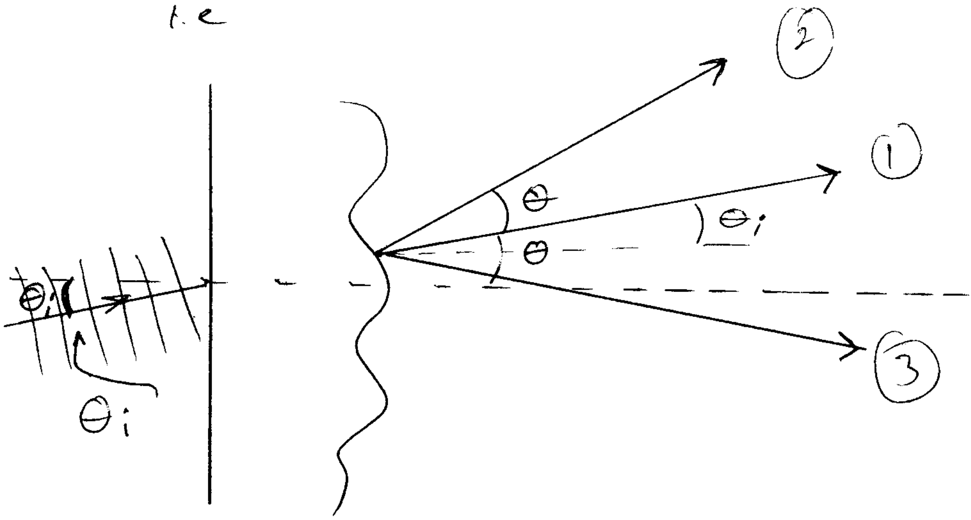


or in other words

$$\sin\theta = \frac{x'}{l} = \frac{\lambda}{\Lambda}$$

(intuitively I hope)
so you can imagine that tilting the input plane wave tilts the outputs

i.e



in Fourier domain

$$g_{out}(x',y') = FT\left(t(x,y) \times f_{in}(x,y)\right)$$

multiplication in space = convolution in freq

$$= FT\left(\text{sine grating} \times e^{-i2\pi \sin\theta_i/\lambda \, x}\right)$$

-or-

modulation in space = shift in freq

$$= \underset{①}{\delta}\left(u - \frac{\sin\theta_i}{\lambda}\right) + \underset{②}{\delta}\left(u - \frac{1}{\Lambda} - \frac{\sin\theta_i}{\lambda}\right) + \underset{③}{\delta}\left(u + \frac{1}{\Lambda} - \frac{\sin\theta_i}{\lambda}\right)$$

remember
$u = x'/\lambda l$
$= \frac{\sin\theta}{\lambda}$

so the angles are given:

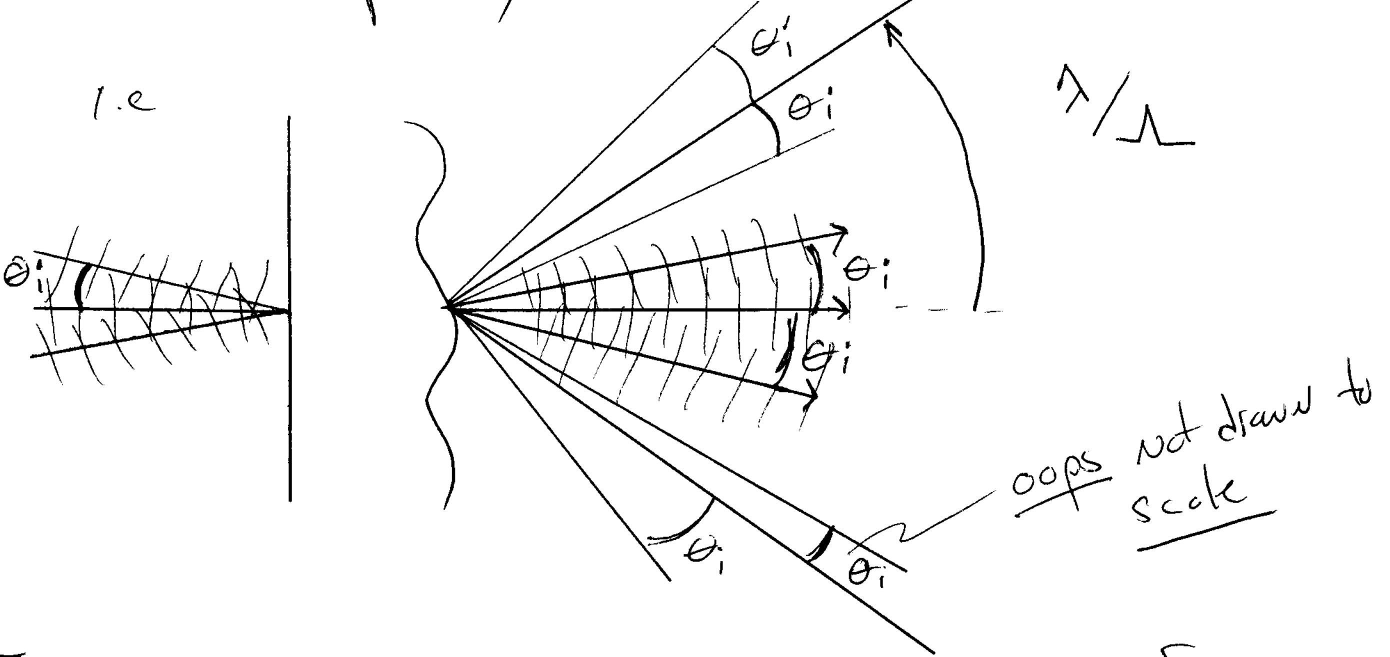
① $\frac{\sin\theta_1}{\lambda} = \frac{\sin\theta_i}{\lambda} \Rightarrow$ in paraxial $\theta_1 \approx \theta_i$

② $\frac{\sin\theta_2}{\lambda} = \frac{1}{\Lambda} + \frac{\sin\theta_i}{\lambda} \Rightarrow \theta_2 = \frac{\lambda}{\Lambda} + \theta_i$

③ $\frac{\sin\theta_3}{\lambda} = \frac{-1}{\Lambda} + \frac{\sin\theta_i}{\lambda} \Rightarrow \theta_3 = -\frac{\lambda}{\Lambda} + \theta_i$

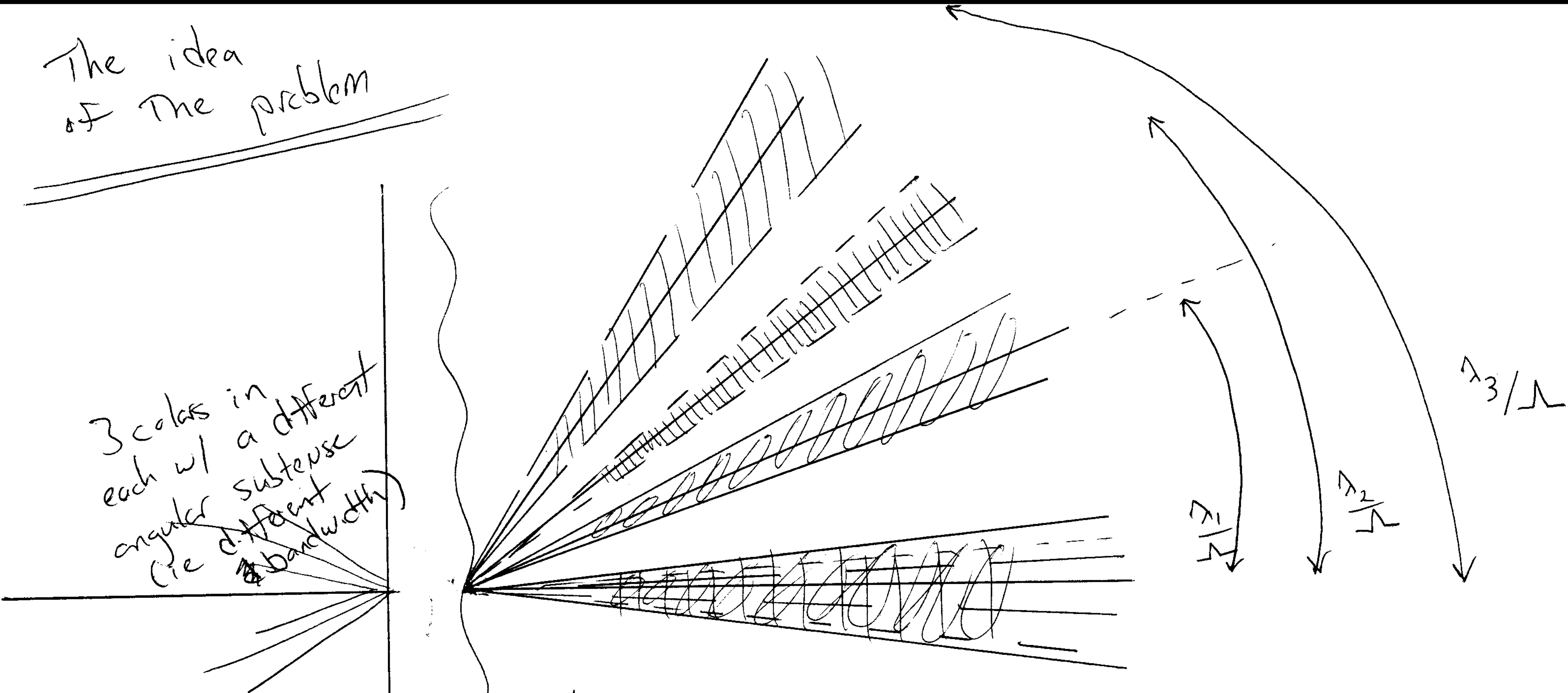
so what does This mean?
IF a sine grating has a cone (limited angular subtense) coming in it will have a similar cone at The output about each primary direction. (diffracted order)

i.e



so we see That if we have cones of multiple colors coming into The grating (each with different $\underline{\lambda}$) They will all travel down The optical axis and Then off at angles That depend on Their $\underline{\underline{\lambda}}$ so what we are trying to achieve is This:

The idea of the problem



Input

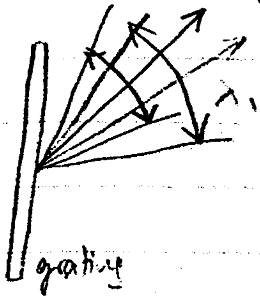
3 colors in, each with a different range of plane waves (angular subtense) (or bandwidth)

Note
$$u_0 = \frac{1}{\Lambda}$$

Seperation out.

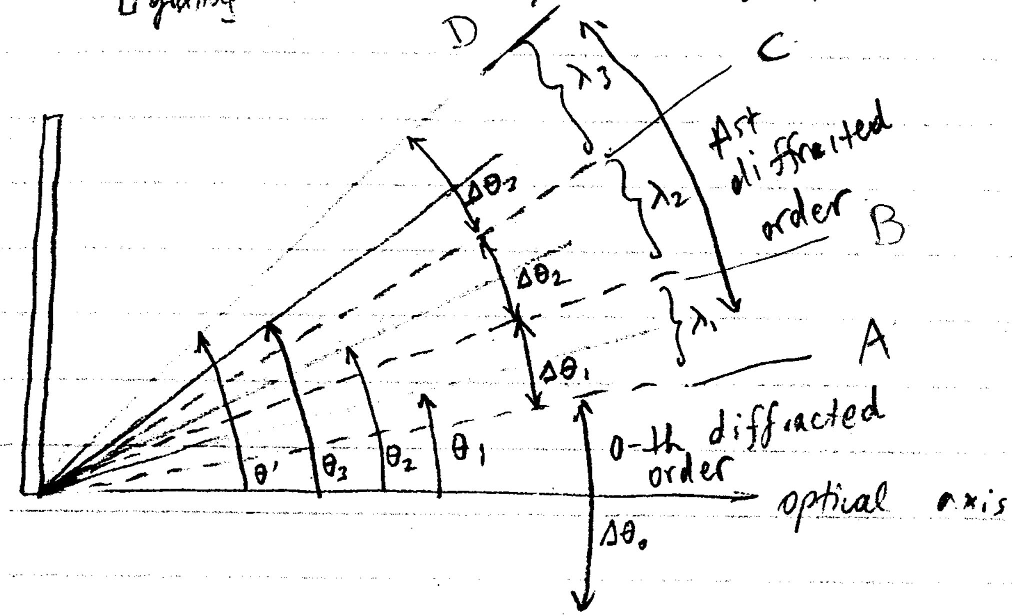
- each range of angles <u>in</u> for each of the 3 colors is centered about its 3 diffraction orders
- and we want to seperate them by making sure that the 3 Bandwidth (angular subtenses of plane waves) and 3 λ's are such that the output bundles don't overlap

(2) a) Broad-band input: 　　　　#4



cannot be separated

∴ spatial spectra must fit within the angle acceptance ranges of lenses L1, L2, L3, respectively



Angular spread of j-th order $(j=1,2,3)$ $\quad \Delta\theta_j = \lambda_j B_j$

0-th order contains all colors $\Rightarrow$ angular spread $\Delta\theta_0 = \max_j \{\lambda_j B_j\}$

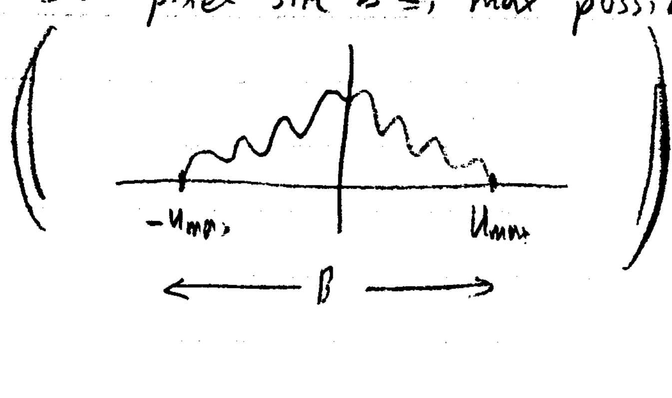
constraints:

$$\begin{cases} A: \theta_1 - \dfrac{\Delta\theta_1}{2} > \dfrac{\Delta\theta_0}{2} \Rightarrow \lambda_1 u_0 - \dfrac{\lambda_1 B_1}{2} \geq \dfrac{1}{2}\max_j \{\lambda_j B_j\} & (1) \\ B: \theta_2 - \dfrac{\Delta\theta_2}{2} > \theta_1 + \dfrac{\Delta\theta_1}{2} \Rightarrow \lambda_2 u_0 - \dfrac{\lambda_2 B_2}{2} \geq \lambda_1 u_0 + \dfrac{\lambda_1 B_1}{2} & (2) \\ C: \theta_3 - \dfrac{\Delta\theta_3}{2} > \theta_2 + \dfrac{\Delta\theta_2}{2} \Rightarrow \lambda_3 u_0 - \dfrac{\lambda_3 B_3}{2} \geq \lambda_2 u_0 + \dfrac{\lambda_2 B_2}{2} & (3) \end{cases}$$

$$D: \theta_3 + \dfrac{\Delta\theta_3}{2} < \theta' \Rightarrow \lambda_3 u_0 + \dfrac{\lambda_3 B_3}{2} \geq \dfrac{1}{2} \quad (4)$$

2×max freq

b) SLM pixel size $b \Rightarrow$ max possible frequency $= \dfrac{1}{2b} \Rightarrow B_1 = B_2 = B_3 = \dfrac{1}{b}$



Substitute ~~into~~ (1) ~~&~~ (4) using

$B_1 = B_3 = \dfrac{1}{b}$ $\quad \lambda_1 = \lambda$ $\quad \lambda_3 = 3\lambda$

(7) b) cont'd

$$(1) \Rightarrow \lambda u_0 - \frac{\lambda}{2b} = \frac{1}{2}\cdot\left(\overbrace{3\lambda\cdot\frac{1}{b}}^{\text{max since } \frac{\lambda}{b} < \frac{\lambda_2}{b} < \frac{3\lambda}{b}}\right) \Rightarrow u_0 = \frac{2}{b}$$

equality ensures orders 0th and 1st @ λ_1 are adjacent
$\Rightarrow \theta_1$ is as small as possible

$$(4) \Rightarrow 3\lambda u_0 + \frac{3\lambda}{2b} = \frac{1}{2} \Rightarrow \frac{15\lambda}{2b} = \frac{1}{2} \Rightarrow b = 15\lambda, \quad u_0 = \frac{2}{15\lambda}$$

equality ensures highest spatial frequency component of order 1st @ λ_3 just misses the θ' constraint

c) Substitute into (2) & (3)

$$(2) \Rightarrow \lambda_2\cdot\frac{2}{15\lambda} - \frac{\lambda_2}{30\lambda} \geq \frac{2\lambda}{15\lambda} + \frac{\lambda}{30\lambda} \Rightarrow \lambda_2 \geq \frac{\frac{2}{15}+\frac{1}{30}}{\frac{2}{15}-\frac{1}{30}}\lambda = \frac{5}{3}\lambda$$

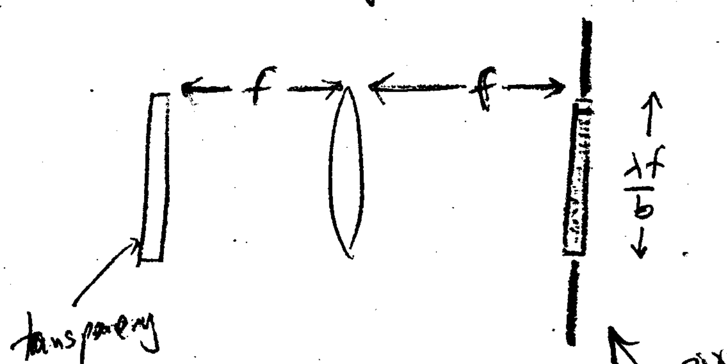
$$(3) \Rightarrow 3\lambda\cdot\frac{2}{15\lambda} - \frac{3\lambda}{30\lambda} \geq \frac{\lambda_2}{5\lambda} + \frac{\lambda_2}{30\lambda} \Rightarrow \lambda_2 \leq 3\,\frac{\frac{2}{15}-\frac{1}{20}}{\frac{2}{15}+\frac{1}{30}}\lambda = \frac{9}{5}\lambda$$

$$\left.\begin{aligned} &\lambda_1 = \lambda \\ &1.67\lambda \leq \lambda_2 \leq 1.8\lambda \\ &\lambda_3 = 3\lambda \end{aligned}\right\}$$

$\leftarrow$ allowable range for λ_2.

d) Sharp edges $\Rightarrow B_{1,2,3} = \infty$ because e.g. rect $\to$ sinc with freq. components extending all over to infinity. So colors mix.

BUT



filters out sharp edge information $\Rightarrow$

$\Rightarrow$ results b, c apply!